Gadau Journal of Pure and Allied Sciences
Gadau J Pure Alli Sci, 1(1): 55-67 (2022)
ISSN: 2955-1722
DOI: https://doi.org/10.54117/gjpas.v1i1.7

On **Ω**-Subgroup

Elijah Eghosa Edeghagba

Department of Mathematical Sciences, Bauchi State University, Gadau, Nigeria.

**Correspondence: edeghagbaelijah@basug.edu.ng; +2348083013747*

Abstract

In the framework of Ω-sets, where Ω is a complete lattice, we present particular kinds of Ω-subgroups: $C_{\mathcal{G}}(A) = (\mathcal{G}, E^{\nu})$, $N_{\mathcal{G}}(A) = (\mathcal{G}, E^{\nu})$ and $Z_{\mathcal{G}}(G) = (\mathcal{G}, E^{\nu})$, based on Ω-group in the language of one binary operation.

Article History
Received: 02/02/2022
Accepted: 28/06/2022
Published: 09/07/2022

Keywords
Ω-set; Ω-groupoid; Ω-group; Ω-subgroup; Ω-Equality; Complete lattice

How to cite this paper: Edeghaba, E.E. (2022). On **Ω**-Subgroup. *Gadau J Pure Alli Sci, 1(1): 55-67*. https://doi.org/10.54117/gjpas.v1i1.7.

1.0 Introduction

1.1 Historical remarks

The concept of fuzzy set was introduction in (Zadeh, 1965). Since then numerous works by researchers have been ongoing in various areas of abstract algebra and related fields in the framework of fuzzy setting. Notably, this concept was quickly adapted by (Goguen, 1967), who introduced the notion of L-fuzzy set. Sanchez (1976) and (Brown, 1971), then generalized the concept of fuzzy sets, in which case the unit interval on the real line used by (Zadeh, 1965) was replace by a suitable partially ordered set (lattice) as the codomain of the fuzzy membership function.

Fuzzy groups and fuzzy semigroups as related concept were introduced and investigation early in the beginning of fuzzy era. The first appearance of fuzzy group was by (Rosenfeld, 1971) . Thereafter, there were many other papers, see the monograph, (Mordeson *et al., 2005*), and references cited there. For fuzzy semigroups, see monograph, (Malik *et al.*, 2003) which gave a comprehensive overview of all the results up to 2003. Among numerous results, we can mention paper (Filep, 1992) where the structure of fuzzy subgroups of a group was investigated, paper (Wen-Xiang and De-Gang, (1994)) where it was shown that a fuzzy subgroupoid of a group need not be a fuzzy group.

In this research the topic investigated is some algebraic aspects of Ω-valued algebraic structures, with focus on groups. Ω is a complete lattice.

Our research originates in fuzzy structures and in Ω-sets. Ω-sets, as an intention for modeling intuitionistic logic, appeared 1979, in the paper (Fourman and Scott, 1979). An Ω-set is a nonempty set A equipped with an Ω-valued equality E, with truth values in a complete Heyting algebra Ω. E is considered to be a symmetric and transitive function from A^2 to Ω. Both theories are well related in their basic ideals since they both deal the notion of belonging in set-theoretic and logical sense. We may also quickly mention that Ω-set in its interpretation is not totally equivalent to that of fuzzy. In fuzzy the notion of subset is generalized by a function, while in Ω-set approach we deal with the so-called "partial elements", where E is not reflexive. Ω-sets have been further applied to non-classical

Journal of the Faculty of Science, Bauchi State University Gadau, Nigeria
Mathematical Sciences

predicate logics, and also to theoretical foundations of the fuzzy set theory (Gottwald, 2006), (Höhle, 2007). Dealing with Ω-structures we use Ω-sets and in our approach Ω is a complete lattice (not necessarily a Heyting algebra). The main reason for this membership values structure is that it allows the use of cut-sets as a tool appearing in the fuzzy set theory. In this setting, main algebraic and set-theoretic notions and their properties can be generalized from their classical origin to the lattice-valued framework (Klir and Yuan, 1995). So we deal also with lattice-valued structures and Ω-sets as basic objects . These were developed within the fuzzy set theory

Since (Goguen, 1967) replaced the unit interval with a complete lattice , this approach has been widely used for dealing with algebraic topics (see Di Nola and Gerla, 1987), then also (Šešelja and Tepavčević, 1993, 1994)), and with the lattice-valued topology starting with (Höhle and Šostak,1999) and many others. In the recent decades, along with the development of the fuzzy logic, a complete lattice as a membership (truth values) structure is often replaced by a complete residuated lattice (see e.g., (Bělohlávek, 2002)). But then the cut structures do not keep algebraic properties satisfied on the basic fuzzy structure.

As a generalization of the classical equality we use lattice-valued equality. This was introduced into fuzzy mathematics by Höhle in his paper (Höhle, 1988), and then several authors have used it in the investigation of fuzzy functions and fuzzy algebraic structures notably among them are: (Demirci, 2003), (Bělohlávek and Vychodil, 2006) and others.

Identities for lattice-valued structures with the fuzzy equality used in this work were introduced in (Šešelja and Tepavčević, 2009) and developed in (Budimirović *et al.*, 2012, 2013, 2014, 2016). Recently, several applications of it have appeared in publication (Edeghagba *et al.*, 2017, 2019), (Bleblou *et al.*, 2018) and (Krapeža *et al.*, 2019). Although similar notion first appeared in (Bělohlávek and Vychodil, 2006). In this framework, an identity holds if the corresponding lattice-theoretic formula is fulfilled. What is new in this approach is that an identity may hold on a lattice-valued algebra, while the underlying classical algebra does not satisfy the analogue classical identity.

In (Šešelja and Tepavčević, 2019) the authors introduced and investigated the notion of Ω-group in the language of Ω-groupiod. In this case the basic algebra is a groupoid with single binary operation. The authors proved that this study of Ω-groups can be equivalent to that of Ω-groups in the language of three operation as presented in (Šešelja and Tepavčević, 2019). In a quick followup to the work presented in (Šešelja and Tepavčević, 2019) the notion of subgroup was introduced in (Edeghagba and Muhammad, 2022) were it was investigated and showed that the lattice-valued identities which holds in an Ω-group also hold in the Ω-subgroup and the notion of Ω-centralizer, Ω-normalize and Ω-center of an Ω-subset of an Ω-group was introduced.

In the present work we investigate and proved that the sets of Ω-centralizers, Ω-normalizers and Ω-centers of an Ω-subset of an Ω- group form Ω-subgroup of the Ω-group , like in the classical group theory.

2.0 Preliminaries

In this section, we give some notations, definitions and propositions which will be needed in this paper.

2.1 Algebras

$\mathcal{G} = (G,*)$ is an algebra in the language with a single binary operation, called $groupoid$.

A $neutralelement$ in a groupoid $\mathcal{G} = (G,*)$ is $x \in G$, such that for every $x \in G$, $x * e = e * x = x$.

A groupoid $\mathcal{G} = (G,*)$ is said to be a $semigroup$ if it fulfills the $associative\ identity$:

$$x * (y * z) \approx (x * y) * z.$$

A $monoid$, G is a semigroup with a neutral element. For $x \in G$ its $inverse$ denoted by $x^{-1} \in G$ is referred to as $inverseelement$, of x in a monoid G, such that

$$x * x^{-1} = x^{-1} * x = e.$$

The inverse of every element $x \in G$ is unique.

A $group$, $\mathcal{G} = (G,*)$ is a monoid in which every element possesses an (unique) inverse.

The following formula defines a neutral element and inverses:

$$(\exists z)(\forall x)(x * z = z * x = x \wedge (\exists y)(x * y = y * x = z)): \qquad (1)$$

in addition to the associative identity. In universal algebra a group can equivalently be defined as an algebra with three operations, $(G;*;x^{-1};e)$: binary operation $*$, unary operation x^{-1} and a nullary operation, constant, e, so that the following identities hold:

$$x * (y * z) \approx (x * y) * z; \qquad (2)$$

$$x * e \approx x; e * x \approx x; \qquad (3)$$

$$x * x^{-1} \approx e; x^{-1} * x \approx e: \qquad (4)$$

Observe that the only identity in the definition of a group as a groupoid $\mathcal{G} = (G,*)$ is associativity.

In this language properties of the neutral element and inverses, as a formula (1), contain existential quantifiers. In the language with three operations, all three formulas, (2, 3, 4), defining a group are identities. Still, these notions describing the structure of a group in different languages are equivalent, as follows.

Proposition 2.1 *If $\mathcal{G} = (G,*)$ is a group as a groupoid with the neutral element e and inverse x^{-1} for $x \in G$, then $(G;*;^{-1};e)$ is a group in the language with three operations. Conversely, if $(G;*;-1;e)$ is a group in the corresponding language, then also the groupoid $\mathcal{G} = (G,*)$ is a group.*

2.2 Ω-valued functions and relations

An **Ω-valued function** μ on a nonempty set A is a mapping $\mu: A \to \Omega$, where $(\Omega, \leq)$ is a complete lattice. This notion can be related to fuzzy set on A. If μ and ν are Ω-valued functions on A, then ν is said to be a **fuzzy subset** of μ, if for all $x \in A$ $\nu(x) \leq \mu(x)$.

For $p \in \Omega$, a **cut set** or a p-**cut** of an Ω-valued function $\mu: A \to \Omega$ is a subset μ_p of A which is the inverse image of the principal filter in Ω, generated by p:

$$\mu_p = \mu^{-1}(\uparrow(p)) = \{x \in X | \mu(x) \geq p\}.$$

An **Ω-valued** (binary) **relation R on A** is an Ω-valued function on A^2, i.e., it is a mapping $R: A^2 \to \Omega$.

R is **symmetric** if

$$R(x,y) = R(y,x) \quad for\ all \quad x,y \in A; \tag{5}$$

R is **transitive** if

$$R(x,y) \geq R(x,z) \wedge R(z,y) \quad for\ all \quad x,y,z \in A. \tag{6}$$

Observe that an Ω-valued symmetric and transitive relation R on A fulfills the **strictness** property (see (Höhle, 2007)):

$$R(x,y) \leq R(x,x) \wedge R(y,y). \tag{7}$$

Likewise we say an Ω-valued symmetric and transitive relation R on A satisfies the separation property if the following holds:

$$R(x,y) = R(x,x) = R(y,x) \neq 0 \quad implies \quad x = y. \tag{8}$$

Therefore, R is said to be separated if 8 holds.

We now consider the connection of the above notion with Ω-valued relations on Ω-set.

Suppose $\mu: A \to \Omega$ is an Ω-valued function on A and $R: A^2 \to \Omega$ an Ω-valued relation on A. If for all $x, y \in A$ the following holds:

$$R(x,y) \leq \mu(x) \wedge \mu(y). \tag{9}$$

then we say that R is an Ω-valued relation on μ.

Next, we consider the notion of (weak) reflexiveness of R. An Ω-valued relation R on $\mu: A \to \Omega$ is said to be reflexive on μ or μ-reflexive if

$$R\ (x,x) = \mu\ (x)\ for\ every\ x \in\ A. \tag{10}$$

A symmetric and transitive Ω-valued relation R on A, which is reflexive on $\mu: A \to \Omega$ is an Ω-valued equivalence on μ.

Clearly, an Ω-valued equivalence R on μ fulfills the strictness property (7).

Furthermore, if R is an Ω-valued equivalence on μ, which is separated according to (8), then we say that R is an Ω-valued equality on μ.

For an algebra $\mathcal{A} = (A, F)$, an Ω-valued function $\mu: A \to \Omega$ is said to be **compatible over $\mathcal{A}$** if μ is not constantly equal to 0, and which fulfils the following: For any operation f from F with arity greater than 0, $f: A^n \to A, n \in \mathbb{N}$, for all $a_1, \dots, a_n \in A$, we have

$$\bigwedge_{i=1}^{n} \mu(a_i) \leq \mu(f(a_1, \dots, a_n)), \tag{11}$$

and for a nullary operation, $c \in F$,

$$\mu(c) = 1. \tag{12}$$

Likewise, an Ω-valued relation $R: A^2 \to \Omega$ on an algebra $\mathcal{A} = (A, F)$ is **compatible** with the operations in F if the following two conditions holds: for every n-ary operation $f \in F$, for all $a_1, \dots, a_n, b_1, \dots, b_n \in A$, and for every constant (nullary operation) $c \in F$

$$\bigwedge_{i=1}^{n} R(a_i, b_i) \leq R(f(a_1, \dots, a_n), f(b_1, \dots, b_n)); \tag{13}$$

$$R(c,c) = 1. \tag{14}$$

2.3 Ω-set

The following is defined in (Fourman and Scott, 1979) and then adopted to a fuzzy framework in (Budimirović *et al.*, 2016).

An **Ω-set** is a pair (A, E), where A is a nonempty set, and E is a symmetric and transitive Ω-valued relation on A, which may fulfill the separation property (8) if indicated. But we note that in this work the separation property is needed for most of the result.

Consequently, for an Ω-set (A, E), we denote by μ the Ω-valued function on A, defined by

$$\mu(x) := E(x,x). \tag{15}$$

Then μ is said to be *determined* by E. This enable us in generalizing the notion of subset of an Ω-set (A, E). Clearly, by the strictness property (7), E is an Ω-valued relation on μ, namely, it is an Ω-valued equality on μ. That is why we say that in an Ω-set (A, E), E is an Ω-valued equality.

Lemma 2.2 *If (A, E) is an Ω-set and $p \in \Omega$, then the cut E_p is an equivalence relation on the corresponding cut μ_p of μ.*

2.4 Ω-algebra

Let $\mathcal{A} = (A, F)$ be an algebra and $E: A^2 \to \Omega$ an Ω-valued equality on A, which is compatible with the operations in F. Then we say that $(\mathcal{A}, E)$ is an Ω-algebra. Algebra $\mathcal{A}$ is the underlying algebra of $(\mathcal{A}, E)$.

Proposition 2.3 *(Budimirović et al., 2016) Let $(\mathcal{A}, E)$ be an Ω-algebra. Then the following hold: (i)* The function $\mu: A \to \Omega$ determined by E ($\mu(x) = E(x, x)$ for all $x \in A$), is compatible over A. (ii) For every $p \in \Omega$, the cut μ_p of μ is a subalgebra of $\mathcal{A}$, and (iii) For every $p \in \Omega$, the cut E_p of E is a congruence relation on μ_p.

2.5 Ω-subalgebra of an Ω-algebra

For reference on the results in this section see (Edeghagba *et al.*, 2017). Let $(\mathcal{A}, E)$ be an Ω-algebra, and (A, E_1) an Ω-subset of (A, E). Then E_1 is a symmetric and transitive Ω-relation on A, fulfilling for all $x, y \in A$

$$E_1(x, y) = E(x, y) \wedge E_1(x, x) \wedge E_1(y, y).$$

Let also E_1 be compatible with the operations in $\mathcal{A}$. Obviously, $(\mathcal{A}, E_1)$ is an Ω-algebra and we say that it is an Ω-subalgebra of the Ω-algebra $(\mathcal{A}, E)$.
The following is obvious.

Proposition 2.4 *If $(\mathcal{A}, E_1)$ is an Ω-subalgebra of an Ω-algebra $(\mathcal{A}, E)$, and $\mu_1: A \to \Omega$ is the Ω-valued function on A defined by $\mu_1(x) := E_1(x, x)$, then μ_1 is compatible over $\mathcal{A}$, i.e., it fulfills* (11).
An Ω-subalgebra $(\mathcal{A}, E_1)$ of $(\mathcal{A}, E)$ fulfills all the identities that the latter does, as follows.
Theorem 2.5 *Let Let $(\mathcal{A}, E_1)$ be an Ω-subalgebra of an Ω-algebra $(\mathcal{A}, E)$. If $(\mathcal{A}, E)$ satisfies the set Σ of identities, then also $(\mathcal{A}, E_1)$ satisfies all the identities in Σ.*

3.0 Ω-groups and Ω-subgroups

3.1 Ω-group: as Ω-groupoid

For reference on the results in this section see (Šešelja and Tepavčević, 2019).

Remark 3.1 *In both languages, the associative property is equivalent to the fulfillment of the corresponding identity in the framework of Ω-algebras:*

$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leq E(x * (y * z), (x * y) * z), \quad (16)$$

Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be an Ω-groupoid, where the underlying algebra is a groupoid $\mathcal{G} = (G, *)$. Then $\overline{\mathcal{G}}$ is a strict Ω-**group** if it is associative in the sense of (16) and the following hold:
There is $e \in G$ such that for every $x \in G$

$$\mu(x) \leq \mu(e) \wedge E(e * x, x) \wedge E(x * e, x) \quad (17)$$

and

$$\textit{there is } x' \in G \textit{ such that } \mu(x) \leq \mu(x') \wedge E(x * x', e) \wedge E(x' * x, e) \quad (18)$$

Proposition 3.2 *Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be an Ω-groupoid which is a strict Ω-group, then the following hold: (i) if $e \in G$ is a neutral element in $\overline{\mathcal{G}}$, then for all $x \in G$ $\mu(e) \geq \mu(x)$.*
(ii) A neutral element e is unique and this is also the neutral element in the underlying groupoid $(G, *)$.

Proposition 3.3 *Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be an Ω-groupoid which is a strict Ω-group, then the following hold for every $x \in G$: (i) $(x')' = x$. (ii) $\mu(x) = \mu(x')$. (iii) Inverse element x' is unique.*
The new result shows the equivalence of the two approaches to Ω-group.

Theorem 3.4 *Let $\mathcal{L}$ be a language with a binary operation $*$, unary operation $^-$ and a nullary operation e. If $((G, *, ^-, e), E)$ is an Ω-group in this language, then also $((G, *), E)$ is an Ω-groupoid which is a strict group. Conversely, if $((G, *), E)$ is an Ω-groupoid which is a strict group where e is a neutral element, x^{-1} is the inverse of $x \in G$ then $((G, *, ^-, e), E)$ is an Ω-group in the language $\mathcal{L}$.*

3.2 Ω-subgroup: as Ω-subgroupoid

For reference on the results in this section see (Edeghagba and Muhammad, 2022).

Proposition 3.5 *Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a strict Ω-group, where $\mathcal{G} = (G, *)$ is a classical groupoid and $\hat{\mathcal{G}} = (\mathcal{G}, \hat{E})$ an Ω-subgroupoid of $\overline{\mathcal{G}}$. Therefore, if $\hat{\mathcal{G}} = (\mathcal{G}, \hat{E})$ is an Ω-subgroup of $\overline{\mathcal{G}}$, then the following holds: There is $e \in \overline{\mathcal{G}}$ such that for all $x \in \overline{\mathcal{G}}$*

$$\hat{\mu}(x) \leq \hat{\mu}(e) \wedge \hat{E}(x * e, x) \wedge \hat{E}(e * x, x) \quad (19)$$

there is $x' \in G$

$$\hat{\mu}(x) \leq \hat{\mu}(x') \wedge \hat{E}(x * x', e) \wedge \hat{E}(x' * x, e) \quad (20)$$

Definition 3.6 *Let $A = (G, E^1)$ to be an Ω-subset of $\overline{\mathcal{G}}$ The we have the following definitions*
1) Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a strict Ω-group , where $\mathcal{G} = (G, *)$ is a classical groupoid and $A = (G, E^1)$ an Ω-subset of $\overline{\mathcal{G}}$. An element $x \in \overline{\mathcal{G}}$ is a centralizer of A if for every $a \in A$ the following holds

$$\mu^1(a) \wedge \mu(x) \leq E(xax', a). \quad (21)$$

2) Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a strict Ω-group , where $\mathcal{G} = (G,*)$ is a classical groupoid and $A = (G, E^1)$ be an Ω-subset of $\overline{\mathcal{G}}$. An element $x \in \overline{\mathcal{G}}$ is a normalizer of A if for all $a \in A$ there exist $b \in A$ such that

$$\mu^1(a) \wedge \mu^1(b) \wedge \mu(x) \le E(xax', b) \qquad (22)$$

3) Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a strict Ω-group , where $\mathcal{G} = (G,*)$ is a classical groupoid. An element $x \in \overline{\mathcal{G}}$ is at the center of $\overline{\mathcal{G}}$ if for all $a \in G$ the following holds

$$\mu(a) \wedge \mu(x) = E(xa, ax) \qquad (23)$$

Proposition 3.7 *Let $A = (G, E^1)$ be an Ω-subset of the (strict) Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$. Then the following hold* i the neutral element of $\overline{\mathcal{G}}$ centralizes A ii if $x \in \overline{\mathcal{G}}$ centralizes A, then the inverse element of $x \in \overline{\mathcal{G}}$ centralizes A iii if $x, y \in \overline{\mathcal{G}}$ centralizes A, then $xy \in \overline{\mathcal{G}}$ centralizes A

Proposition 3.8 *Let $A = (G, E^1)$ be an Ω-subset of the (strict) Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$. Then the following hold* i the neutral element of $\overline{\mathcal{G}}$ normalizes A ii if $x \in \overline{\mathcal{G}}$ normalizes A, then the inverse element of $x \in \overline{\mathcal{G}}$ normalizes A iii if $x, y \in \overline{\mathcal{G}}$ normalizes A, then $xy \in \overline{\mathcal{G}}$ normalizes A

Proposition 3.9 *Let $A = (G, E^1)$ be an Ω-subset of the (strict) Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$. Then the following hold* i the neutral element of $\overline{\mathcal{G}}$ is at the center of $\overline{\mathcal{G}}$ ii if $x \in \overline{\mathcal{G}}$ is at the center of $\overline{\mathcal{G}}$, then the inverse element of $x \in \overline{\mathcal{G}}$ is at the center of $\overline{\mathcal{G}}$ iii if $x, y \in \overline{\mathcal{G}}$ is at the center of $\overline{\mathcal{G}}$, then $xy \in \overline{\mathcal{G}}$ is at the center of $\overline{\mathcal{G}}$.

4.0 Results

In subsection (3.2) Ω-subgroup of an Ω-group (as presented in (Šešelja and Tepavčević, 2019) was investigated in the case where the underlying algebra is an algebra with one operation. But our aim here is to deal with particular kinds Ω-subgroups of an Ω-group in the language of one operation, as was introduced in (Edeghagba and Muhammad, 2022).

Let $A = (G, E^1)$ be an Ω-subset of the (strict) Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$ and the Ω-relation $E^\nu : G^2 \longrightarrow \Omega$ be defined by

$$E^\nu(x, x) = E(xax', a) \wedge E^1(a', a') \qquad (24)$$

for which

$$E^\nu(x, y) = E(x, y) \wedge E^1(a, a) \qquad (25)$$

for $a, x, y \in G$.

We consider ν to be Ω-valued function on G which is an Ω-valued subset of the Ω-valued function μ.

Proposition 4.1 *Let E^ν be a symmetry and transitive Ω-relation on G as given by equation (24) above and fulfilling $E^\nu \le E$. Then the following holds:*

$$E^\nu(x, y) = E^\nu(x, x) \wedge E^\nu(y, y) \wedge E(x, y) \qquad (26)$$

for $a, x, y \in G$

Proof. Clearly $E^\nu(x, y) \le E^\nu(x, x) \wedge E^\nu(y, y) \wedge E(x, y)$. By (24), $E^\nu(x, x) = E(xax', a) \wedge E^1(a', a')$ and $E^\nu(y, y) = E(yay', a) \wedge E^1(a', a')$, for $x, y \in G$. Thus

$$E^\nu(x, x) \wedge E^\nu(y, y) = E(xax', a) \wedge E^1(a', a') \wedge E(yay', a) \wedge E^1(a', a') = E(xax', a) \wedge E^1(a', a') \wedge E(yay', a)$$

$$\Rightarrow E^\nu(x, x) \wedge E^\nu(y, y) \le E(xax', yay') \wedge E^1(a', a') \qquad (Transitivity)$$

$$\Rightarrow E^\nu(x, x) \wedge E^\nu(y, y) \wedge E(x, y) \le E(xax', yay') \wedge E(x, y) \wedge E^1(a', a') \le E(xax'x, yay'y) \wedge E^1(a', a') = E(xae, yae) \wedge E^1(a', a') \quad (Inverse\ element\ in\ \overline{\mathcal{G}})$$

$$\Rightarrow E^\nu(x, x) \wedge E^\nu(y, y) \wedge E(x, y) \le E(xa, ya) \wedge E^1(a', a') = E(xa, ya) \wedge E^1(a', a') \wedge E(a', a') \qquad (where\ \ \mu^1(a') \le \mu(a'))$$

$$\Rightarrow E^\nu(x, x) \wedge E^\nu(y, y) \wedge E(x, y) \le E(xaa', yaa') \wedge E^1(a', a') \quad (compatibility)$$

$$\Rightarrow E^\nu(x, x) \wedge E^\nu(y, y) \wedge E(x, y) \le E(xe, ye) \wedge E^1(a', a') = E(x, y) \wedge E^1(a', a') \quad (a'\ inverse\ of\ a\ in\ \overline{\mathcal{G}})$$

$$\Rightarrow E^\nu(x, x) \wedge E^\nu(y, y) \wedge E(x, y) \le E(x, y) \wedge E^1(a', a') = E(x, y) \wedge E^1(a', a') \wedge E^1(a, a) = E(x, y) \wedge E^1(a, a) \qquad (where\ \ \mu^1(a) = \mu^1(a'))$$

$$\Rightarrow E^\nu(x, x) \wedge E^\nu(y, y) \wedge E(x, y) \le E(x, y) \wedge E^1(a, a) = E^\nu(x, y) \quad (By\ \ (25))$$

$$\Rightarrow E^\nu(x, x) \wedge E^\nu(y, y) \wedge E(x, y) \le E^\nu(x, y).$$

Hence $E^\nu(x, y) = E^\nu(x, x) \wedge E^\nu(y, y) \wedge E(x, y)$ as required.

Clearly Proposition 4.1 shows that E^ν is a restriction of E to the nonempty Ω-subset ν of μ (where μ is determine by E). Therefore, the pair (G, E^ν) is an Ω-set as presented in Proposition 4.1 and hence an Ω-subset of the Ω-set (G, E).

Corollary 4.2 *Let $A=(G,E^1)$ be an Ω-subset of the Ω-set (G,E) and (G,E^ν) an Ω-subset of the Ω-set (G,E) as given by equation (24) above. Then*

$$E^\nu(x,x)\wedge E^1(a,a)\le E(xax',a) \tag{27}$$

for $x,a\in G$

Proof. Let $x,a\in G$ and by equation (24), $E^\nu(x,x)=E(xax',a)\wedge E^1(a',a')$, thus

$$E^\nu(x,x)=E(xax',a)\wedge E^1(a',a')$$
$$\Rightarrow E^\nu(x,x)\wedge E^1(a,a)=E(xax',a)\wedge E^1(a',a')\wedge E^1(a,a)\le E(xax',a)$$
$$\wedge E(a',a')\wedge E(a,a)\le E(xax',a)\wedge E(a'a,a'a) \qquad (compatibility)$$
$$\Rightarrow E^\nu(x,x)\wedge E^1(a,a)\le E(xax',a)\wedge E(e,e) \quad (a' \text{ inverse of } a \text{ in } \overline{\mathcal{G}})$$
$$\Rightarrow E^\nu(x,x)\wedge E^1(a,a)\le E(xax',a).$$

Hence $E^\nu(x,x)\wedge E^1(a,a)\le E(xax',a)$ as required.

We understand that $\mu^1: G\longrightarrow\Omega$ is an Ω-valued function on G defined by $\mu^1(x):=E^1(x,x)$ and the Ω-valued function $\nu: G\longrightarrow\Omega$ on G is defined by $\nu(x):=E^\nu(x,x)$. Hence we rewrite equation (27) as

$$\nu(x)\wedge\mu^1(a)\le E(xax',a) \tag{28}$$

Remark 4.3 *Observe that the Ω-valued function ν on G which is an Ω-valued subset of the Ω-valued function μ fulfills equation (21), therefore the Ω-set (G,E^ν) will be referred to as an Ω-centralizer of the Ω-set $A=(G,E^1)$.*

Like in the crips classical group theory the set of centralizers of a given subset of the a group forms a subgroup of the group. Therefore, the next results gives this analogy for an Ω-group.

Proposition 4.4 *Let $A=(G,E^1)$ be an Ω-subset of the Ω-set (G,E) and (G,E^ν) an Ω-centralizer of the Ω-set $A=(G,E^1)$. If E^ν is compatible with the operation defined in the (strict) Ω-group $\overline{\mathcal{G}}=(\mathcal{G},E)$, then $C_{\mathcal{G}}(A)=(\mathcal{G},E^\nu)$ is an Ω-subgroup of $\overline{\mathcal{G}}=(\mathcal{G},E)$.*

Proof. First we show the existence of inverse element for each element in $C_{\mathcal{G}}(A)$. Let $x,a\in G$, x' be the inverse of x in $\overline{\mathcal{G}}=(\mathcal{G},E)$ and e the neutral element in $\overline{\mathcal{G}}=(\mathcal{G},E)$ and by equation (24), $E^\nu(x,x)=E(xax',a)\wedge E^1(a',a')$ and $E^\nu(x',x')=E(x'ax,a)\wedge E^1(a',a')$. If $q=E(x'ax,a)\wedge E(a',a')$ then $E(x'ax,a)\ge q$. Therefore, $(x'ax,a)\in E_q$ implying

$$[x'ax]_{E_q}=[a]_{E_q}$$
$$\Rightarrow [x']_{E_q}[a]_{E_q}[x]_{E_q}=[a]_{E_q}$$
$$\Rightarrow [x]_{E_q}=[a']_{E_q}[x]_{E_q}[a]_{E_q}$$
$$\Rightarrow (x,a'xa)\in E_q$$
$$\Rightarrow E(x,a'xa)\ge q.$$

similarly, If $q=E(xax',a)\wedge E(a',a')$ then $E(xax',a)\ge q$. Therefore, $(xax',a)\in E_q$ implying

$$[xax']_{E_q}=[a]_{E_q}$$
$$\Rightarrow [x]_{E_q}[a]_{E_q}[x']_{E_q}=[a]_{E_q}$$
$$\Rightarrow [x']_{E_q}=[a']_{E_q}[x']_{E_q}[a]_{E_q}$$
$$\Rightarrow (x',a'x'a)\in E_q$$
$$\Rightarrow E(x',a'x'a)\ge q.$$

Thus, $E^\nu(x,x)=E(xax',a)\wedge E^1(a',a')=E(xax',a)\wedge E^1(a',a')\wedge E^1(a,a)$ and $E^\nu(x',x')=E(x'ax,a)\wedge E^1(a',a')=E(x'ax,a)\wedge E^1(a',a')\wedge E^1(a,a)$. Therefore

$$E^\nu(x,x)\wedge E^\nu(x',x')=E(xax',a)\wedge E^1(a',a')\wedge E^1(a,a)\wedge E(x'ax,a)\wedge E^1(a',a')$$
$$\wedge E^1(a,a)=E^1(a',a')\wedge E(xax',a)\wedge E^1(a,a)\wedge E^1(a',a')\wedge E(x'ax,a)\wedge E^1(a,a)$$
$$\le E(a',a')\wedge E(xax',a)\wedge E^1(a,a)\wedge E(a',a')\wedge E(x'ax,a)\wedge E^1(a,a)$$
$$\Rightarrow E^\nu(x,x)\wedge E^\nu(x',x')\le E(a'xax',a'a)\wedge E^1(a,a)\wedge E(a'x'ax,a'a)\wedge E^1(a,a)$$
$$(compatibility)$$
$$\Rightarrow E^\nu(x,x)\wedge E^\nu(x',x')\le E(a'xax',e)\wedge E^1(a,a)\wedge E(a'x'ax,e)\wedge E^1(a,a)$$
$$(a' \text{ inverse of } a \text{ in } \overline{\mathcal{G}}).$$

let $q\ge p=E^\nu(x,x)\wedge E^\nu(x',x')$ then $E(a'xax',e)\wedge E^1(a,a)\wedge E(a'x'ax,e)\wedge E^1(a,a)\ge p$. Therefore, $(a'xax',e)\in E_p$ and $(a'x'ax,e)\in E_p$ implying

$$[a'xax']_{E_p}=[e]_{E_p}$$

$$\Rightarrow [a']_{E_p}[x]_{E_p}[a]_{E_p}[x']_{E_p} = [e]_{E_p}$$
$$\Rightarrow [a'xa]_{E_p}[x']_{E_p} = [e]_{E_p}$$
$$\Rightarrow [x]_{E_p}[x']_{E_p} = [e]_{E_p} \qquad (q \ \geq \ p)$$
$$\Rightarrow (xx', e) \in E_p$$
$$\Rightarrow E(xx', e) \geq p.$$

and

$$[a'x'ax]_{E_p} = [e]_{E_p}$$
$$\Rightarrow [a']_{E_p}[x']_{E_p}[a]_{E_p}[x]_{E_p} = [e]_{E_p}$$
$$\Rightarrow [a'x'a]_{E_p}[x]_{E_p} = [e]_{E_p}$$
$$\Rightarrow [x']_{E_p}[x]_{E_p} = [e]_{E_p} \qquad (q \ \geq \ p)$$
$$\Rightarrow (x'x, e) \in E_p$$
$$\Rightarrow E(x'x, e) \geq p.$$

Thus

$$E^{\nu}(x,x) \wedge E^{\nu}(x',x') \leq E(xx',e) \wedge E^1(a,a) \wedge E(x'x,e) \wedge E^1(a,a)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(x',x') \wedge E^{\nu}(x',x') \leq E(xx',e) \wedge E^1(a,a) \wedge E(x'x,e) \wedge E^1(a,a) \wedge E^{\nu}(x',x')$$
$$\Rightarrow E^{\nu}(x,x) \leq E(xx',e) \wedge E^1(a,a) \wedge E(x'x,e) \wedge E^1(a,a) \wedge E^{\nu}(x',x')$$
$$(where \ \ E^{\nu}(x',x') \ \geq \ E^{\nu}(x,x))$$
$$\Rightarrow E^{\nu}(x,x) \leq E^{\nu}(xx',e) \wedge E^{\nu}(x'x,e) \wedge E^{\nu}(x',x') \qquad (by \ \ equation(25))$$
$$\Rightarrow \nu(x) \leq \nu(x') \wedge E^{\nu}(xx',e) \wedge E^{\nu}(x'x,e). \qquad (E^{\nu}(x,x) := \nu(x) \ )$$

Hence $\nu(x) \leq \nu(x') \wedge E^{\nu}(xx',e) \wedge E^{\nu}(x'x,e)$.
Next we show the existence of the the neutral element in $C_{\mathcal{G}}(A)$. By equation (24), $E^{\nu}(x,x) = E(xax',a) \wedge E^1(a',a')$ and $E^{\nu}(e,e) = E(eae',a) \wedge E^1(a',a')$. If $q = E(x'ax,a) \wedge E(a',a')$ then $E(x'ax,a) \geq q$. Therefore, $(x'ax,a) \in E_q$ implying
$[xax']_{E_q} = [a]_{E_q}$.
similarly, If $q = E(eae',a) \wedge E(a',a')$ then $E(eae',a) \geq q$. Therefore, $(eae',a) \in E_q$ implying
$[eae']_{E_q} = [a]_{E_q}$.
Thus, $E^{\nu}(x,x) = E(xax',a) \wedge E^1(a',a') = E(xax',a) \wedge E^1(a',a') \wedge E^1(a,a)$ and $E^{\nu}(e,e) = E(eae',a) \wedge E^1(a',a') = E(eae',a) \wedge E^1(a',a') \wedge E^1(a,a)$. Therefore,

$$E^{\nu}(x,x) \wedge E^{\nu}(e,e) = E(xax',a) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E(eae',a) \wedge E^1(a',a') \wedge E^1(a,a) = E(xax',a) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E^1(a',a') \wedge E(eae',a) \wedge E^1(a,a) \leq E(xax',a) \wedge E(a',a') \wedge E^1(a,a) \wedge E(a',a') \wedge E(eae',a) \wedge E^1(a,a)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(e,e) \leq E(xax'a',aa') \wedge E^1(a,a) \wedge E(a'eae',aa') \wedge E^1(a,a)$$
$$(compatibility)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(e,e) \leq E(xax'a',e) \wedge E^1(a,a) \wedge E(a'eae',e) \wedge E^1(a,a)$$
$$(a' \ \ inverse \ \ of \ \ a \ \ in \ \ \overline{\mathcal{G}})$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(e,e) \leq E(xax'a',e) \wedge E(x,x) \wedge E^1(a,a) \wedge E(x,x) \wedge E(a'eae',e) \wedge E^1(a,a) \leq E(xax'a'x,ex) \wedge E^1(a,a) \wedge E(xa'eae',xe) \wedge E^1(a,a)$$
$$(compatibility)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(e,e) \leq E(xax'a'x,x) \wedge E^1(a,a) \wedge E(xa'eae',x) \wedge E^1(a,a)$$
$$(e \ \ the \ \ neutral \ \ element \ \ in \ \ \overline{\mathcal{G}})$$

let $q \geq p = E^{\nu}(x,x) \wedge E^{\nu}(e,e)$ then $E(xax'a'x,x) \wedge E^1(a,a) \wedge E(xa'eae',x) \wedge E^1(a,a) \geq p$. Therefore, $(xax'a'x,x) \in E_p$ and $(xa'eae',x) \in E_p$ implying

$$[xax'a'x]_{E_p} = [x]_{E_p}$$
$$\Rightarrow [xax']_{E_p}[a']_{E_p}[x]_{E_p} = [x]_{E_p}$$
$$\Rightarrow [a]_{E_p}[a']_{E_p}[x]_{E_p} = [x]_{E_p} \qquad (q \ \geq \ p)$$
$$\Rightarrow [aa']_{E_p}[x]_{E_p} = [x]_{E_p}$$
$$\Rightarrow [e]_{E_p}[x]_{E_p} = [x]_{E_p} \qquad (Inverse \ \ element)$$
$$\Rightarrow [ex]_{E_p} = [x]_{E_p}$$
$$\Rightarrow (ex,x) \in E_p$$

$$\Rightarrow E(ex,x) \geq p.$$

and

$$[xa'eae']_{E_p} = [x]_{E_p}$$
$$\Rightarrow [x]_{E_p}[a']_{E_p}[eae']_{E_p} = [x]_{E_p}$$
$$\Rightarrow [x]_{E_p}[a']_{E_p}[a]_{E_p} = [x]_{E_p} \qquad (q \;\geq\; p)$$
$$\Rightarrow [x]_{E_p}[a'a]_{E_p} = [x]_{E_p}$$
$$\Rightarrow [x]_{E_p}[e]_{E_p} = [x]_{E_p} \qquad (Inverse\;\; element)$$
$$\Rightarrow [xe]_{E_p} = [x]_{E_p}$$
$$\Rightarrow (xe,x) \in E_p$$
$$\Rightarrow E(xe,x) \geq p.$$

Thus

$$E^{\nu}(x,x) \wedge E^{\nu}(e,e) \leq E^1(ex,x) \wedge E(a,a) \wedge E^1(xe,x) \wedge E(a,a)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(e,e) \wedge E^{\nu}(e,e) \leq E(ex,x) \wedge E^1(a,a) \wedge E(xe,x) \wedge E^1(a,a) \wedge E^{\nu}(e,e)$$
$$\Rightarrow E^{\nu}(x,x) \leq E(ex,x) \wedge E^1(a,a) \wedge E(xe,x) \wedge E^1(a,a) \wedge E^{\nu}(e,e)$$
$$(where\;\; E^{\nu}(e,e) \geq \;E^{\nu}(x,x))$$
$$\Rightarrow E^{\nu}(x,x) \leq E^{\nu}(ex,x) \wedge E^{\nu}(xe,x) \wedge E^{\nu}(e,e) \qquad (by\;\; equation(25))$$
$$\Rightarrow \nu(x) \leq \nu(e) \wedge E^{\nu}(ex,x) \wedge E^{\nu}(xe,x). \qquad (\; where\;\; E^{\nu}(x,x) := \nu(x)\;)$$

Hence $\nu(x) \leq \nu(e) \wedge E^{\nu}(ex,x) \wedge E^{\nu}(xe,x)$.
We have prove that $C_{\mathcal{G}}(A) = (\mathcal{G}, E^{\nu})$ is an Ω-subgroup of $\overline{\mathcal{G}} = (\mathcal{G}, E)$.

Let $A = (G, E^1)$ be an Ω-subset of the (strict) Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$ and $E^{\nu}: G^2 \longrightarrow \Omega$ be defined by
$E^{\nu}(x,x) = E(xax',b) \wedge E^1(a',a') \wedge E^1(b',b')$ (29)
for which
$E^{\nu}(x,y) = E(x,y) \wedge E^1(a,a) \wedge E^1(b,b)$ (30)
for $a,b,x,y \in G$
We consider ν be Ω-valued function on G which is an Ω-valued subset of the Ω-valued function μ.

Proposition 4.5 *Let E^{ν} be a symmetry and transitive Ω relation on G as given by equation 24 above and fulfilling $E^{\nu} \leq E$. Then the following holds:*

$E^{\nu}(x,y) = E^{\nu}(x,x) \wedge E^{\nu}(y,y) \wedge E(x,y)$ (31)

Proof. Clearly $E^{\nu}(x,y) \leq E^{\nu}(x,x) \wedge E^{\nu}(y,y) \wedge E(x,y)$. By equation (29) $E^{\nu}(x,x) = E(xax',b) \wedge E^1(a',a') \wedge E^1(b',b')$ and $E^{\nu}(y,y) = E(yay',b) \wedge E^1(a',a') \wedge E^1(b',b')$, for $x,y \in G$. Thus

$$E^{\nu}(x,x) \wedge E^{\nu}(y,y) = E(xax',b) \wedge E^1(a',a') \wedge E^1(b',b') \wedge E(yay',b) \wedge E^1(a',a') \wedge E^1(b',b') = E(xax',b) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E(yay',b) \wedge E^1(b',b') \wedge E^1(b,b)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(y,y) \leq E(xax',b) \wedge E(yay',b) \wedge E(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b) \leq E(xax',yay') \wedge E(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b) \qquad (Transitivity)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(y,y) \wedge E(x,y) \leq E(xax',yay') \wedge E(x,y) \wedge E(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b) \leq E(xax'x,yay'y) \wedge E(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b) \qquad (Compatibility)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(y,y) \wedge E(x,y) \leq E(xae,yae) \wedge E(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b) \qquad (Inverse\;\; of\;\; element\;\; in\;\; \overline{\mathcal{G}})$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(y,y) \wedge E(x,y) \leq E(xa,ya) \wedge E(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(y,y) \wedge E(x,y) \leq E(xaa',yaa') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b) \qquad (compatibility)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(y,y) \wedge E(x,y) \leq E(xe,ye) \wedge E^1(a,a) \wedge E^1(b,b) = E(x,y) \wedge E^1(a,a) \wedge E^1(b,b) \qquad (a'\;\; inverse\;\; of\;\; a\;\; in\;\; \overline{\mathcal{G}})$$

$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(y,y) \wedge E(x,y) \leq E(x,y) \wedge E^{1}(a,a) \wedge E^{1}(b,b) = E^{\nu}(x,y) \quad (By\ (30))$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(y,y) \wedge E(x,y) \leq E^{\nu}(x,y).$$
Hence $E^{\nu}(x,y) = E^{\nu}(x,x) \wedge E^{\nu}(y,y) \wedge E(x,y)$ as required.

Clearly Proposition 4.5 shows that E^{ν} is a restriction of E to the nonempty Ω-subset ν of μ (where μ is determine by E). Therefore, the pair (G, E^{ν}) is an Ω-set as presented in Proposition 4.5 and hence an Ω-subset of the Ω-set (G, E).

Corollary 4.6 *Let $A = (G, E^{1})$ be an Ω-subset of the Ω-set (G, E) and (G, E^{ν}) an Ω-subset of the Ω-set (G, E) as given above. Then*

$$E^{\nu}(x,x) \wedge E^{1}(a,a) \wedge E^{1}(b,b) \leq E(xax',b) \tag{32}$$
for $x, a \in G$

Proof. Let $x, a \in G$ and by equation (29), $E^{\nu}(x,x) = E(xax',b) \wedge E^{1}(a',a') \wedge E^{1}(b',b')$, thus
$$E^{\nu}(x,x) = E(xax',b) \wedge E^{1}(a',a') \wedge E^{1}(b',b')$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{1}(a,a) \wedge E^{1}(b,b) = E(xax',b) \wedge E^{1}(a',a') \wedge E^{1}(b',b') \wedge E^{1}(a,a) \wedge E^{1}(b,b) \leq E(xax',b) \wedge E(a',a') \wedge E(a,a) \wedge E(b',b') \wedge E(b,b) \leq E(xax',a) \wedge E(a'a,a'a) \wedge E(b'b,b'b) \qquad (compatibility)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{1}(a,a) \wedge E^{1}(b,b) \leq E(xax',b) \wedge E(e,e) \wedge E(e,e) \quad (Inverse\ element\ in\ \overline{\mathcal{G}})$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{1}(a,a) \wedge E^{1}(b,b) \leq E(xax',b).$$
Hence $E^{\nu}(x,x) \wedge E^{1}(a,a) \wedge E^{1}(b,b) \leq E(xax',b)$ as required.

We understand that $\mu^{1}: G \longrightarrow \Omega$ is an Ω-valued function on G defined by $\mu^{1}(x) := E^{1}(x,x)$ and the Ω-valued function $\nu: G \longrightarrow \Omega$ on G is defined by $\nu(x) := E^{\nu}(x,x)$. Hence we rewrite equation (32) as
$$\nu(x) \wedge \mu^{1}(a) \wedge \mu^{1}(b) \leq E(xax',b) \tag{33}$$

Remark 4.7 *Observe that the Ω-valued function ν on G which is an Ω-valued subset of the Ω-valued function μ fulfills equation (21), therefore the Ω-set (G, E^{ν}) will be referred to as an Ω-normalizer of the Ω-set $A = (G, E^{1})$.*
Like in the crips classical group theory the set of elements of a group that normalizes a given subset of the a group forms a subgroup of the group. Therefore, the next result gives this analogy for an Ω-group.

Proposition 4.8 *Let $A = (G, E^{1})$ be an Ω-subset of the Ω-set (G, E) and (G, E^{ν}) an Ω-normalizers of the Ω-set $A = (G, E^{1})$. If E^{ν} is compatible with the operation defined in the (strict) Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$, then $N_{\mathcal{G}}(A) = (\mathcal{G}, E^{\nu})$ is an Ω-subgroup of $\overline{\mathcal{G}} = (\mathcal{G}, E)$.*

Proof. First we show the existence of inverse element for each element in $N_{\mathcal{G}}(A)$. Let $x, a, b \in G$, x' be the inverse of x in $\overline{\mathcal{G}} = (\mathcal{G}, E)$ and e the neutral element in $\overline{\mathcal{G}} = (\mathcal{G}, E)$ and by equation (29), $E^{\nu}(x,x) = E(xax',b) \wedge E^{1}(a',a') \wedge E^{1}(b',b')$ and $E^{\nu}(x',x') = E^{\nu}(x'bx,a) \wedge E^{1}(a',a') \wedge E^{1}(b',b')$. If $q = E(x'ax,b) \wedge E(a',a') \wedge E^{1}(b',b')$ then $E(x'ax,b) \geq q$. Therefore, $(x'ax,b) \in E_q$ implying
$$[x'ax]_{E_q} = [b]_{E_q}$$
$$\Rightarrow [x']_{E_q}[a]_{E_q}[x]_{E_q} = [b]_{E_q}$$
$$\Rightarrow [x]_{E_q} = [b]_{E_q}[x]_{E_q}[a']_{E_q}$$
$$\Rightarrow (x, bxa') \in E_q$$
$$\Rightarrow E(x, bxa') \geq q.$$
similarly, If $q = E(xbx',a) \wedge E(a',a')$ then $E(xbx',a) \geq q$. Therefore, $(xbx',a) \in E_q$ implying
$$[xbx']_{E_q} = [a]_{E_q}$$
$$\Rightarrow [x]_{E_q}[b]_{E_q}[x']_{E_q} = [a]_{E_q}$$
$$\Rightarrow [x']_{E_q} = [a]_{E_q}[x']_{E_q}[b']_{E_q}$$
$$\Rightarrow (x', ax'b') \in E_q$$
$$\Rightarrow E(x', ax'b') \geq q.$$

Thus, $E^{\nu}(x,x) = E(xax',b) \wedge E^1(a',a') \wedge E^1(b',b') = E(xax',b) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b)$ and $E^{\nu}(x',x') = E(x'bx,a) \wedge E^1(a',a') \wedge E^1(b',b') = E(x'bx,a) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b)$. Therefore

$$E^{\nu}(x,x) \wedge E^{\nu}(x',x') = E(xax',b) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b) \wedge E(x'bx,a) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b) = E(xax',b) \wedge E^1(b',b') \wedge E^1(b,b) \wedge E^1(a,a) \wedge E^1(a',a') \wedge E(x'bx,a) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b) \leq E(xax',b) \wedge E(b',b') \wedge E^1(b,b) \wedge E^1(a,a) \wedge E(a',a') \wedge E(x'bx,a) \wedge E(a',a') \wedge E^1(a,a) \wedge E(b',b') \wedge E^1(b,b) = E(xax',b) \wedge E(b',b') \wedge E^1(b,b) \wedge E^1(a,a) \wedge E(x'bx,a) \wedge E(a',a') \wedge E^1(a,a) \wedge E^1(b,b)$$

$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(x',x') \leq E(xax'b',bb') \wedge E^1(b,b) \wedge E^1(a,a) \wedge E(x'bxa',aa') \wedge E^1(a,a) \wedge E^1(b,b) \quad (compatibility)$$

$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(x',x') \leq E(xax'b',e) \wedge E^1(b,b) \wedge E^1(a,a) \wedge E(x'bxa',e) \wedge E^1(a,a) \wedge E^1(b,b) \quad (Inverse\ element\ in\ \overline{\mathcal{G}}).$$

let $q \geq p = E^{\nu}(x,x) \wedge E^{\nu}(x',x')$ then $E(xax'b',e,e) \wedge E^1(a,a) \wedge E^1(b,b) \wedge E(x'bxa',e) \wedge E^1(a,a) \wedge E^1(b,b) \geq p$. Therefore, $(xax'b',e) \in E_p$ and $(x'bxa',e) \in E_p$ implying

$$[xax'b']_{E_p} = [e]_{E_p}$$
$$\Rightarrow [x]_{E_p}[a]_{E_p}[x']_{E_p}[b']_{E_p} = [e]_{E_p}$$
$$\Rightarrow [x]_{E_p}[ax'b']_{E_p} = [e]_{E_p}$$
$$\Rightarrow [x]_{E_p}[x']_{E_p} = [e]_{E_p} \quad (q \geq p)$$
$$\Rightarrow (xx',e) \in E_p$$
$$\Rightarrow E(xx',e) \geq p.$$

and

$$[x'bxa']_{E_p} = [e]_{E_p}$$
$$\Rightarrow [x']_{E_p}[b]_{E_p}[x]_{E_p}[a']_{E_p} = [e]_{E_p}$$
$$\Rightarrow [x']_{E_p}[bxa']_{E_p} = [e]_{E_p}$$
$$\Rightarrow [x']_{E_p}[x]_{E_p} = [e]_{E_p} \quad (q \geq p)$$
$$\Rightarrow (x'x,e) \in E_p$$
$$\Rightarrow E(x'x,e) \geq p.$$

Thus

$$E^{\nu}(x,x) \wedge E^{\nu}(x',x') \leq E(xx',e) \wedge E^1(a,a) \wedge E^1(b,b) \wedge E(x'x,e) \wedge E^1(a,a) \wedge E^1(b,b)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(x',x') \wedge E^{\nu}(x',x') \leq E(xx',e) \wedge E^1(a,a) \wedge E^1(b,b) \wedge E(x'x,e) \wedge E^1(a,a) \wedge E^1(b,b) \wedge E^{\nu}(x',x')$$
$$\Rightarrow E^{\nu}(x,x) \leq E(xx',e) \wedge E^1(a,a) \wedge E^1(b,b) \wedge E(x'x,e) \wedge E^1(a,a) \wedge E^1(b,b) \wedge E^{\nu}(x',x') \quad (where\ E^{\nu}(x',x') \geq E^{\nu}(x,x))$$
$$\Rightarrow E^{\nu}(x,x) \leq E^{\nu}(xx',e) \wedge E^{\nu}(x'x,e) \wedge E^{\nu}(x',x') \quad (by\ equation(30))$$
$$\Rightarrow \nu(x) \leq \nu(x') \wedge E^{\nu}(xx',e) \wedge E^{\nu}(x'x,e). \quad (E^{\nu}(x,x) := \nu(x))$$

Hence $\nu(x) \leq \nu(x') \wedge E^{\nu}(xx',e) \wedge E^{\nu}(x'x,e)$.

Next we show the existence of the neutral element in $N_{\mathcal{G}}(A)$. By equation (29), $E^{\nu}(x,x) = E(xax',b) \wedge E^1(a',a') \wedge E^1(b',b')$ and $E^{\nu}(e,e) = E(eae',b) \wedge E^1(a',a') \wedge E^1(b',b')$. If $q = E(x'ax,b) \wedge E(a',a')$ then $E(x'ax,b) \geq q$. Therefore, $(x'ax,b) \in E_q$ implying

$$[xax']_{E_q} = [b]_{E_q}.$$

similarly, If $q = E(eae',b) \wedge E(a',a') \wedge E(b',b')$ then $E(eae',b) \geq q$. Therefore, $(eae',b) \in E_q$ implying

$$[eae']_{E_q} = [b]_{E_q}.$$

Thus, $E^{\nu}(x,x) = E(xax',b) \wedge E^1(a',a') \wedge E^1(b',b') = E(xax',a) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b)$ and $E^{\nu}(e,e) = E(eae',a) \wedge E^1(a',a') \wedge E^1(b',b') = E(eae',a) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b)$. Therefore

$$E^{\nu}(x,x) \wedge E^{\nu}(e,e) = E(xax',b) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b) \wedge E(eae',b) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b) = E(xax',b) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b) \wedge E(eae',b) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E^1(b',b') \wedge E^1(b,b) \leq E(xax',b) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E(b',b') \wedge E^1(b,b) \wedge E(eae',b) \wedge E^1(a',a') \wedge E^1(a,a) \wedge E(b',b') \wedge E^1(b,b)$$

$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(e,e) \leq E(xax'b',bb') \wedge E^1(a,a) \wedge E^1(b,b) \wedge E(eae'b',bb') \wedge E^1(a,a) \wedge E^1(b,b) \qquad (compatibility)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(e,e) \leq E(xax'b',e) \wedge E^1(a,a) \wedge E^1(b,b) \wedge E(eae'b',e) \wedge E^1(a,a) \wedge E^1(b,b) \qquad (b' \ inverse \ of \ b \ in \ \overline{\mathcal{G}})$$

let $q \geq p = E^{\nu}(x,x) \wedge E^{\nu}(e,e)$ then $E(xax'b'x,x) \wedge E^1(a,a) \wedge E^1(b,b) \wedge E(xeae'b',x) \wedge E^1(a,a) \wedge E^1(b,b) \geq p$. Therefore, $(xax'b'x,x) \in E_p$ and $(xeae'b',x,x) \in E_p$ implying

$$[xax'b'x]_{E_p} = [x]_{E_p}$$
$$\Rightarrow [xax']_{E_p}[b']_{E_p}[x]_{E_p} = [x]_{E_p}$$
$$\Rightarrow [b]_{E_p}[b']_{E_p}[x]_{E_p} = [x]_{E_p} \qquad (q \geq p)$$
$$\Rightarrow [bb']_{E_p}[x]_{E_p} = [x]_{E_p}$$
$$\Rightarrow [e]_{E_p}[x]_{E_p} = [x]_{E_p} \qquad (Inverse \ element)$$
$$\Rightarrow [ex]_{E_p} = [x]_{E_p}$$
$$\Rightarrow (ex,x) \in E_p$$
$$\Rightarrow E(ex,x) \geq p.$$

and

$$[xeae'b']_{E_p} = [x]_{E_p}$$
$$\Rightarrow [x]_{E_p}[eae']_{E_p}[b']_{E_p} = [x]_{E_p}$$
$$\Rightarrow [x]_{E_p}[b]_{E_p}[b']_{E_p} = [x]_{E_p} \qquad (q \geq p)$$
$$\Rightarrow [x]_{E_p}[bb']_{E_p} = [x]_{E_p}$$
$$\Rightarrow [x]_{E_p}[e]_{E_p} = [x]_{E_p} \qquad (Inverse \ element)$$
$$\Rightarrow [xe]_{E_p} = [x]_{E_p}$$
$$\Rightarrow (xe,x) \in E_p$$
$$\Rightarrow E(xe,x) \geq p.$$

Thus

$$E^{\nu}(x,x) \wedge E^{\nu}(e,e) \leq E(ex,x) \wedge E^1(a,a) \wedge E^1(b,b) \wedge E(xe,x) \wedge E^1(a,a) \wedge E^1(b,b)$$
$$\Rightarrow E^{\nu}(x,x) \wedge E^{\nu}(e,e) \wedge E^{\nu}(e,e) \leq E(ex,x) \wedge E^1(a,a) \wedge E^1(b,b) \wedge E(xe,x) \wedge E^1(a,a) \wedge E^1(b,b) \wedge E^{\nu}(e,e)$$
$$\Rightarrow E^{\nu}(x,x) \leq E(ex,x) \wedge E^1(a,a) \wedge E^1(b,b) \wedge E(xe,x) \wedge E^1(a,a) \wedge E^1(b,b) \wedge E^{\nu}(e,e) \quad (where \ E^{\nu}(e,e) \geq \ E^{\nu}(x,x))$$
$$\Rightarrow E^{\nu}(x,x) \leq E^{\nu}(ex,x) \wedge E^{\nu}(xe,x) \wedge E^{\nu}(e,e) \qquad (by \ equation (30))$$
$$\Rightarrow \nu(x) \leq \nu(e) \wedge E^{\nu}(ex,x) \wedge E^{\nu}(xe,x). \qquad (E^{\nu}(x,x) := \nu(x) \ )$$

Hence $\nu(x) \leq \nu(e) \wedge E^{\nu}(ex,x) \wedge E^{\nu}(xe,x)$.

We have prove that $N_{\mathcal{G}}(A) = (\mathcal{G}, E^{\nu})$ is an Ω-subgroup of $\overline{\mathcal{G}} = (\mathcal{G}, E)$.

Of course $N_{\mathcal{G}}(A) = (\mathcal{G}, E^{\nu})$ is an Ω-subgroupoid of $\overline{\mathcal{G}} = (\mathcal{G}, E)$.

Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be an (strict) Ω-group and $E: G^2 \longrightarrow \Omega$ be defined by

$$E^{\nu}(x,x) = E(xa,xa) \wedge E(a',a') \qquad (34)$$

for which

$$E^{\nu}(x,y) = E(x,y) \wedge E^1(a,a) \qquad (35)$$

for $a, x, y \in G$

We consider ν be Ω-valued function on G which is an Ω-valued subset of the Ω-valued function μ.

Proposition 4.9 *Let E^{ν} be a symmetry and transitive Ω relation on G as given by equation 34 above and fulfilling $E^{\nu} \leq E$. Then the following holds:*

$$E^{\nu}(x,y) = E^{\nu}(x,x) \wedge E^{\nu}(y,y) \wedge E(x,y) \qquad (36)$$

Proof. Proof analogous to the proof of proposition (4.1).

Clearly proposition 4.9 shows that E^{ν} is a restriction of E to the nonempty Ω-subset ν of μ (where μ is determine by E). Therefore, the pair (G, E^{ν}) is an Ω-set as presented in Proposition 4.9 and hence an Ω-subset of the Ω-set (G, E).

Corollary 4.10 *Let (G, E) be an Ω-set and (G, E^{ν}) an Ω-subset of the Ω-set (G, E) as given above. Then*

$$E^{\nu}(x,x) \wedge E(a,a) \leq E(xax',a) \qquad (37)$$

for $x, a \in G$

Proof. Proof analogous to the proof of proposition (4.2).

The Ω-valued function $\nu: G \longrightarrow \Omega$ on G is defined by $\nu(x) := E^{\nu}(x, x)$. Hence we rewrite equation (27) as

$$\nu(x) \wedge \mu(a) \leq E(xa, xa) \qquad (38)$$

Remark 4.11 *Observe that the Ω-valued function ν on G which is an Ω-valued subset of the Ω-valued function μ fulfills equation (21), therefore the Ω-set (G, E^{ν}) will be referred to as an Ω-center of the (strict) Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$.*

Like in the crips classical group theory the set of elements in the center of a given group forms a subgroup of the group. Therefore, the next results gives this analogy for an Ω-group.

Proposition 4.12 *Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ a (strict) Ω-group and (G, E^{ν}) an Ω-center of $\overline{\mathcal{G}} = (\mathcal{G}, E)$. If E^{ν} is compatible with the operation defined in $\overline{\mathcal{G}} = (\mathcal{G}, E)$, then $Z_{\mathcal{G}}(G) = (\mathcal{G}, E^{\nu})$ is an Ω-subgroup of $\overline{\mathcal{G}} = (\mathcal{G}, E)$.*

Proof. Proof analogous to the proof of proposition (4.4).

5.0 Conclusion

In the framework of Ω-group in the language of Ω-groupoid some particular Ω-subgroup of an Ω-group as in the case of classical group theory generated by Ω-centralizers, Ω-centers and Ω-normalizers of an Ω-subset of an Ω-group are presented and investigated. In this framework our next task is to further our investigation into Ω-normal subgroups, Ω-homomorphisms, Ω-isomrphisms and then Ω-group action and its related concepts.

Declarations

Ethics approval and consent to participate

Not Applicable

Consent for publication

Sole author submission

Availability of data and material

Not Applicable.

Competing interests

Author declare no competing interests.

Funding

There was no funding for the current report.

References

Bělohlávek, R. (2002). *Fuzzy Relational Systems: Foundations and Principles*, Kluwer Academic/Plenum Publishers, New York.

Bělohlávek, R. and Vychodil, V. (2006). *Algebras with fuzzy equalities*, Fuzzy Sets and Systems 157: 161—201.

Bleblou, O.S.A., Šešelja, B. and Tepavčević, A. (2018). *Normal Ω-subgroup* , Faculty of Sciences and Mathematics, University of Nis, Serbia, 32 , (19):6699-6711.

Brown, J. G. (1971). *A Note on Fuzzy Sets*, Information and Control, 18: 32-39.

Budimirović, B., Budimirović, V., Šešelja, B. and Tepavčević, A. (2012). *Fuzzy equational classes*, Fuzzy Systems (FUZZ-IEEE), IEEE International Conference, 1-6.

Budimirović, B., Budimirović, V., Šešelja, B. and Tepavčević, A. (2013). *Fuzzy Equational Classes are Fuzzy Varieties*, Iranian Journal of Fuzzy Systems 10: 1-18.

Budimirović, B., Budimirović, V., Šešelja, B. and Tepavčević, A. (2014). *Fuzzy Identities with Application to Fuzzy Semigroups*, Information Sciences, 266: 148-159.

Budimirović, B., Budimirović, V., Šešelja, B. and Tepavčević, A. (2016). *E-Fuzzy Groups*, Fuzzy Sets and Systems 289: 94-112.

Demirci, M. (2003). *Foundations of fuzzy functions and vague algebra based on many-valued equivalence relations part I: fuzzy functions and their applications, part II: vague algebraic notions, part III: constructions of vague algebraic notions and vague arithmetic operations*, Int. J. General Systems 32 (3): 123-155, 157-175, 177-201.

Di Nola, A and Gerla, G. (1987). *Lattice Valued Algebras*, Stochastica 11: 137—150.

Edeghagba, E. E. and Muhammad, F. U. (2022). *An Introduction to Ω-Subgroup*, Journal of Mathematical Sciences and Optimization: Theory and Applications, In print.

Edeghagba, E.E., Šešelja, B. and Tepavčević, A. (2017). *Congruences and Homomorphisms on Ω- Algebras* , Kybernetika, 53,(5): 892-910.

Edeghagba, E.E., Šešelja, B. and Tepavčević, A. (2019). *Representation Theory for Complete L-Lattices* , J. of Mult.-Valued Logic and Soft Computing, 33 , (6):593-617. 25p.

Filep, L. (1992). *Structure and Constructing of Fuzzy Subgroups of a Group*, Fuzzy Sets Syst. 51: 105-109.

Fourman, M.P and Scott, D.S. (1979). *Sheaves and Logic*, in: M.P. Fourman, C.J. Mulvey D.S. Scott (Eds.), Applications of Sheaves, Lecture Notes in Mathematics, vol. 753, Springer, Berlin, Heidelberg, New York, 302—401.

Goguen, J.A.(1967). *L-fuzzy Sets*, J. Math. Anal. Appl. (18): 145-174.

Gottwald, S. (2006). *Universes of Fuzzy Sets and Axiomatizations of Fuzzy Set Theory, Part II: Category theoretic approaches*, Studia Logica, 84(1): 23-50. 1143-1174.

Höhle, U. (1988). *Quotients with respect to Similarity Relations*, Fuzzy Sets and Systems 27: 31-44.

Höhle, U. (2007). *Fuzzy Sets and Sheaves. Part I: Basic Concepts*, Fuzzy Sets and Systems, (11): 1-58. Klir, G. and Yuan, B. (1995). *Fuzzy Sets and Fuzzy Logic*, Prentice Hall P T R, New Jersey.

Höhle, U. and Šostak, A.P. (1999). *Axiomatic Foundations of Fixed-basis Fuzzy Topology*, Springer US, 123-272.

Krapeža, A., Šešelja, B. and Tepavčević, A. (2019). *Solving Linear Equations in Fuzzy Quasigroups* , Information Sciences, 491: 179-189.

Malik, D.S., Mordeson , J.N. and Kuroki, N. (2003). *Fuzzy Semigroups* Studies in Fuzziness and Soft Computing, vol. 131, Springer-Verlag, Berlin, Heidelberg.

Mordeson, J.N., Bhutani, K.R. and Rosenfeld, A. (2005). *Fuzzy Group Theory* Studies in Fuzziness and Soft Computing, vol. 182, Springer-Verlag, Berlin, Heidelberg.

Rosenfeld, A. (1971). *Fuzzy groups*, J. Math. Anal, 35: 512–517.

Sanchez, E. (1976). *Resolution of Composite Fuzzy Relation Equations*, Information and Control, 30: 38-48.

Šešelja, B and Tepavčević, A. (2009). *Fuzzy Identities*, Proc. of the IEEE International Conference on Fuzzy Systems, 1660—1664.

Šešelja, B. and Tepavčević, A. (1993). *Partially Ordered and Relational Valued Algebras and Congruences*, Review of Research, Faculty of Science, Mathematical Series 23: 273—287.

Šešelja, B. and Tepavčević, A. (1994). *On Generalizations of Fuzzy Algebras and Congruences*, Fuzzy Sets and Systems 65: 85—94.

Šešelja, B. and Tepavčević, A. (2019). *Ω-group in the Language of Ω-groupiod*, Fuzzy Sets and Systems, 397: 152-167.

Wen-Xiang, G and De-Gang, C. (1994). *A Fuzzy Subgroupoid which is not a Fuzzy Group*, Fuzzy Sets Syst. 62: 115–116.

Zadeh, L.A. (1965). Fuzzy Sets, Information and Control, 8, 338-353.