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A differential game of pursuit for an infinite system of simple motion in the plane

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Abstract

In the present paper, we investigate a differential game of pursuit for an infinite system of simple motion in a plane. The control functions of the players satisfy both geometric and integral constraints respectively. In the plane, the game assumes to be completed if the state of the pursuer x_k, $k = 1,2, \ldots$ is directly coincides with that of the evader y_k, $k = 1,2, \ldots$ i.e; $x_k(\xi) = y_k(\xi)$, $k = 1,2, \ldots$ at some time ξ and the evader tries to stop the incidence. In addition to that the strategy of the pursuer with respect to geometric and integral constraints will be constructed. Moreover, a numerical example will be given to illustrate the result.

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1.0 Introduction

With diversification of literature being published on differential games with two different constraints on control functions of players, i.e geometric and integral constraints, these include Chernous'ko, 1992, Egorov, 2004, Ibragimov, 2002, Mamatov, 2008, Satimov and Tukhtasinov, 2006, Tukhtasinov and Mamatov, 2009, Belousov, 2010, Ja'afaru and Ibragimov, 2012, Kuchkarov, 2013, Satimov and Tukhtasinov, 2005, Gafurjan *et al*., 2019, Usman and Gafurjan, 2019, Samatov, 2013, Ibragimov *et al*., 2014, Salami *et al*., 2016, Mamatov and Tukhtasinov, 2009, Tukhtasinov and Mamatov, 2008, Ibragimov and Salami, 2009, Ibragimov *et al*., 2015, Ibragimov and Hussin, 2010, Ibragimov, 2013, Ibragimov *et al*., 2014, Ibragimov and Hasim, 2010 and Ibragimov *et al*., 2014. They have extensively discussed and concentrated only on either pursuit problem or pursuit and evasion differential game problems where either integral, geometric or both constraints are imposed on control functions of the players.

Differential game of one pursuer and one evader with integral constraints studied by (Gafurjan, 2002), occurs on a closed convex subset S of $\mathbb{R}$ and dynamics of the players are described by the equations

$$\dot{x} = \alpha u, x(0) = x_0, \int_0^\infty |u(s)| ds \le \rho^2 \quad (1)$$
$$\dot{y} = \alpha v, y(0) = y_0, \int_0^\infty |v(s)| ds \le \rho^2 \quad (2)$$

In the paper, evasion and pursuit problems were investigated and a formula for an optimal pursuit time was found and optimal strategies of the players were constructed.

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Ibragimov and Yusra (2012) studied evasion differential game described by simple equations of (1) and (2) with the case $\alpha = 1$ which involved many pursuers and one evader in the plane. Each coordinate of the control functions of the players is subjected to integral constraints. Sufficient condition for which evasion to be possible is found, and evader's strategy is constructed based on controls of the pursuers.
In the paper of Alias *et al*. (2016), study a simple motion differential game of many pursuers and one evader described by

$$\dot{x}_i = u_i, \, x_i(0) = x_{i0}, i = 1, \ldots m, y \doteq v, \; y(0) = y_0 \quad (3)$$

In the plane, we give a nonempty closed convex set in the plane, and the pursuers and evader move on this set. They cannot leave this set during the game. Control functions of players are subject to coordinate-wise integral constraints

$$\int_0^\infty |u_{ij}(s)| ds \le \rho_{ij}^2, \int_0^\infty |v_j(s)| ds \le \sigma_j^2 \; j = 1,2, \ldots, \quad (4)$$

the game is said to be completed if $x_i(t_i) = y(t_i)$ at some t_i.
Salimi and Ferrara (2018), studied a simple motion differential game with finite number of pursuers and one evader with integral constraints imposed on control functions of the players in Hilbert Space l_2. The equations of motion are described by (3) above. The authors solved the problem and found the value of the game under the assumption that the energy resource of each pursuer is not necessarily greater than that of the evader, and optimal strategies of the pursuers were also constructed.
Ibragimov (2005) studied a pursuit evasion game of many pursuers and one evader with geometric restrictions on the control functions of the players in Hilbert Space l_2. Ibragimov and Salimi (2009) investigate a differential game with infinitely many inertia players with integral restrictions on the control functions. In both papers, the payoff function of the game is the greatest lower bound of the distances between the evader and the pursuer when the game is finished. The evader's aim is to maximize the payoff, and the pursuer's aim is to minimize it. The authors solved the problem and found the value of the game under the condition that the energy resource of each pursuer is greater than the energy resource of the evader.
The paper of Mehdi and Missimiliano (2019), studied a differential game in which a finite or countable number of pursuers pursue a single evader. The control functions of players satisfy integral constraints in Hilbert Space

$$l_2 = \left\{ \alpha = (\alpha_k), k\epsilon N \, \epsilon R^N : \sum_{k=1}^{\infty} \alpha_k^2 \le \infty \right\}$$

with inner product and norm as $\langle \alpha, \beta \rangle$ and $\|\alpha\|_{l_2} = (\sum_{k=1}^{\infty} \alpha_k^2)^{1/2}$. The period of the game which is denoted as ξ is determined. The farness between the evader and the closest pursuer when the game is finished is the payoff function of the game. In the present paper, we introduce the value of the game and identify optimal strategies of the pursuers.

Ridwan and Badakaya (2019), studied a pursuit differential game problem with countable number of pursuers and one evader formulated by simple motion (3) in Hilbert space. Control functions of some finite number of pursuers are subject to integral constraints, while that of the remaining pursuers and evader are subject to geometric constraint. Sufficient conditions for completion of pursuit in two different theorems are presented.

In this present work, we concentrate more on differential game of pursuit for an infinite system of simple motion as of the paper Gafurjan (2022) but with the case of $\alpha = 1$, where control functions of the players are subjected to integral and geometric constraints, unlike the work of Alias (2016) where they investigated the case of only integral constraint in the plane and the paper Rilwan and Badakaya (2018) where case of integral and geometric constraints in Hilbert Space respectively.
This paper is organized based on sections. The first section dealt with the formulation of the problems and vital definitions which were used in the subsequent sections. In the second section, a differential game of pursuit with different forms of constraints is presented which were used to obtain the main result of this paper. In the third section, a numerical example was provided to illustrate the result. Finally, the conclusion was derived from the results.

2.0 Formulation of the Problems

In R, we study a differential game of pursuit described by infinite system of simple motio in xy plane.

$$\begin{cases} (P): \dot{x}_k = u_k(t), & x_k(0) = x_k^0 \\ (E): \dot{y}_k = v_k(t), & y_k(0) = y_k^0 \end{cases} \; x_k^0 \ne y_k^0, k = 1,2, \ldots, t\epsilon[0, \xi] \quad (5)$$

Where ξ be a given number, $x^0 = (x_1^0,\ x_2^0, \dots)$, $y^0 = (y_1^0,\ y_2^0, \dots)$ and $u = (u_1,\ u_2, \dots)$, $v = (v_1,\ v_2, \dots)$ are the control parameters of the pursuer (P) and the evader (E) respectively.

Definition 1 A measurable functions$u(t) = (u_1(t),\ u_2(t), \dots)$, and $v(t) = (v_1(t),\ v_2(t), \dots)$ $0 \le t \ge \xi$ are called admissible controls of the pursuer (P) and the evader (E) respectively, if they satisfy the following inequalities

$$\sum_{k=1}^{\infty} \int_0^{\xi} |u_k(s)|^2 \le \rho^2, \sum_{k=1}^{\infty} \int_0^{\xi} |v_k(s)|^2 \le \sigma^2 \tag{6}$$

$$\sum_{k=1}^{\infty} |u_k(t)|^2 \le \rho^2,\ \sum_{k=1}^{\infty} |v_k(s)|^2 \le \sigma^2 \tag{7}$$

Where ρ and σ are nonnegative numbers (Usman *et al*., 2018).

Remark 1 Let denote $\psi_1(\rho)$ and $\psi_2(\rho)$ be the set of all admissible controls of the pursuer (P) with respect to geometric and integral constraints respectively.

Definition 2 Also let denote the admissible control of the pursuer (P) and the evader (E) be u$(.)\epsilon\ \psi(\rho)$ and v$(.)\epsilon\ \psi(\sigma)$ respectively be chosen, the corresponding motion $x(.)$ and y$(.)$ of the players defined as

$$x(t) = (x_1(t),\ x_2(t), \dots), y(t) = (y_1(t),\ y_2(t), \dots),$$

$$x_k(t) = x_k^0 + \int_0^{\xi} u_k(t)dt, y_k(t) = y_k^0 + \int_0^{\xi} v_k(t)dt,$$

are called the solution of the infinite system (5).

Remark 2 In a plane, the infinite system of simple motion (5) in which $u(.) = (u_1(.), u_2(.), \dots)$, and $v(.) = (v_1(.), v_2(.), \dots)$ satisfied inequality (6) and (7) are called differential game Ω_1 and Ω_2 respectively (Usman *et al*., 2018).

Definition 3 A function $u(t,x,y,v) = (u_1(t,x,y,v),\ u_2(t,x,y,v), \dots)0 \le t \ge \xi$ is referred to as the strategy of pursuer(P), with respect to geometric constraint if for any admissible control $v = v(t)\ 0 \le t \ge \xi$ of evader (E) and $u(.), v(.)\epsilon\ \psi_1(\rho)$.

Remark 3 In similar passion, the strategy of the pursuer (ρ) with respect to integral constraint will be defined and $u(.),\ v(.)\epsilon\ \psi_2(\rho)$.

Definition 4 We say that pursuit can be completed for the time $\xi > 0$ from initial state $x^0 = (x_1^0,\ x_2^0, \dots)$, $y^0 = (y_1^0,\ y_2^0, \dots)$ in differential game Ω_1 and Ω_2, if there exists a strategy $u(t,x,y,v)$ of pursuer (P) such that, for any admissible control $v(t)\ 0 \le t \ge \xi$ of evader (E), the equality

$$x_k(t) = y_k(t)\ 0 \le t \ge \xi, k = 1,2, \dots$$

holds.

Problem 1 The problem is to find a condition on the initial position$x^0 = (x_1^0,\ x_2^0, \dots)$, $y^0 = (y_1^0,\ y_2^0, \dots)$ which guarantee pursuit in differential game Ω_1 and Ω_2 for some time ξ and construct the strategies of the pursuer$(P)$with the case of different forms of constraints.

3.0 A Differential Game of Pursuit with Differential Form of Constraints

In this section, we study pursuit differential game described by equation (5). In the game, the goal of pursuer P is to ensure the equality$x_k(\xi) = y_k(\xi)$ for some ξ, and that of the evader's goal E is opposite. Then let denote the control and energy resource of the pursuer be ρ and denote the control and energy resource of the evader be σ. If $\rho > \sigma$, then we must construct a suitable strategy for the pursuer to ensure $x_k(\xi) = y_k(\xi)$, for some ξ and for any admissible control of the evader.

Before the main result, we must consider the following lemma

Lemma 1 *If $\rho_1 > \sigma_1$, then there exist a strategy of the pursuer such that $x_1(\theta) = y_1(\theta)$, and $x_2(\theta) = y_2(\theta)$ at some time θ in differential games Ω_1 and Ω_2 respectively.*

Proof: We set the strategy of pursuer with respect to geometric constraint (see definition 3 with $k = 1$), as

$$u_1(t)=\frac{y_{10}-x_{10}}{\sqrt{\theta}}+v_1(t), 0\le t\le\sqrt{\theta} \tag{8}$$

For $t>0, u_1(t)=0$

In similar way one can set the strategy for which satisfies integral constraint (see definition 3 with $k=1$), as

$$u_2(t)=\frac{y_{20}-x_{20}}{\theta}+v_2(t), 0\le t\le\sqrt{\theta} \tag{9}$$

For $t>0, u_2(t)=0$, where

$$\theta=\frac{|y_{10}-x_{10}|^2}{(\rho_1-\sigma_1)^2}$$

Now we show that strategy (8) admissible. Using Minkowski sum inequality and definition 1 with $k=1$, and the fact that $u_1(t)=\psi_1(\rho)$ we have

$$(|u_1(t)|^2)^{1/2}=\left(\left|\frac{y_{10}-x_{10}}{\sqrt{\theta}}+v_1(t)\right|^2\right)^{1/2}$$

$$\le\left(\left|\frac{y_{10}-x_{10}}{\sqrt{\theta}}\right|^2\right)^{1/2}+(|v_1(t)|^2)^{1/2}$$
$$\le\rho_1-\sigma_1+\sigma_1=\rho_{1,}$$

Next is to prove that the strategy (9) is also admissible. From the fact that $u_2(t)\epsilon\ \psi_2(\rho)$, using minkwoski integral inequality and definition 1 with$k=1$, we get

$$\left(\int_0^\theta|u_2(t)|^2\,dt\right)^{1/2}=\left(\int_0^\theta\left|\frac{y_{20}-x_{20}}{\theta}+v_2(t)\right|^2\,dt\right)^{1/2}$$
$$\le\left(\int_0^\theta\left|\frac{y_{20}-x_{20}}{\theta}\right|^2dt\right)^{1/2}+\left(\int_0^\theta|v_2(t)|^2dt\right)^{1/2}$$
$$\le\left(\left|\frac{y_{20}-x_{20}}{\theta}\right|\right)^{1/2}+\sigma_1$$
$$=\rho_1-\sigma_1+\sigma_1=\rho_1$$

Consequently, the strategy (8) and (9) are admissible.

Next we show that $x_1(\sqrt{\theta})=y_1(\sqrt{\theta})$ at some$\sqrt{\theta}$. In accordance with the strategy (8) and definition 2 and 4 with $k=1$, we have

$$x_1(\sqrt{\theta})=x_{10}+\int_0^{\sqrt{\theta}}\left(\frac{y_{10}-x_{10}}{\sqrt{\theta}}+v_1(t)\right)dt=y_{10}+\int_0^{\sqrt{\theta}}v_1(t)dt=y_1(\sqrt{\theta})$$

Similarly, we prove the equality $x_2(\theta)=y_2(\theta)$ at some θ holds. Using admissible strategy (9) definition 2 and 4 with $k=1$, indeed

$$x_2(\theta)=x_{10}+\int_0^{\theta}\left(\frac{y_{20}-x_{20}}{\theta}+v_2(t)\right)dt=y_{20}+\int_0^{\theta}v_2(t)dt=y_2(\theta)$$

Unlike the Lemma of Ibragimov [28] in the above proof, the inequality $\rho\le\sigma$ does not need to be satisfied. Hence, the proof of the lemma is completed with the case $k=1$.

In a situation where the system is extended to an infinite system of simple motion i.e,; $k=1,2,\ldots$

Let

$$\xi_1=\sum_{k=1}^{\infty}\frac{|y_{10}-x_{10}|^2}{(\rho_1-\sigma_1)^2}, \rho>\sigma, k=1,2,\ldots \tag{10}$$

Of course $\xi_1\le\xi_2$.

We state the main theorem for which the pursuer satisfies geometric constraint.

Theorem 1 *If* $\rho > \sigma$*, then* $x_k(\xi) = y_k(\xi)$ $k = 1,2,\dots,\xi > 0$ *from initial position* $x^0 = (x_1^0,\ x_2^0, \dots)$, $y^0 = (y_1^0,\ y_2^0, \dots)$ *in differential games* Ω_1 *for some time* ξ

Proof: To solve the stated theorem, we divided the solution into three steps, first is to define the strategy, second is to validate the constructed strategy by showing its admissibility and prove that the game is completed.

A_1: We construct the strategy of the pursuer(P). Strategy is constructed based on control of the pursuer. In this part each complement of the control functions of players satisfies geometric constraint and the control parameters are in the right hand sides of equation in additive form.

$$U_k(t,x,y,v) = \begin{cases} \dfrac{y_k^0 - x_k^0}{\sqrt{\xi_1}} + v_k(t) & 0 \le t \le \xi_1 \\ v_k(t), & t > \xi_1, \end{cases} \quad k = 1,2,\dots \tag{11}$$

A_2: Now, we show that strategy (11) is admissible. It should be noted that, in the pursuit game, the evader use arbitrary admissible controls and the pursuer use strategy (11) and $U(.,v(.))\epsilon\, \psi_1(\rho)$. Indeed

$$\sum_{k=1}^{\infty} |U_k(t,x,y,v)|^2 = \sum_{k=1}^{N} |U_k(t,x,y,v)|^2 + \sum_{k=N+1}^{\infty} |U_k(t,x,y,v)|^2$$

Using definition 3 and strategy (11), we get

$$\sum_{k=1}^{\infty} |U_k(t,x,y,v)|^2 = \sum_{k=1}^{\infty} \left| \frac{y_k^0 - x_k^0}{\sqrt{\xi_1}} + v_k(t) \right|^2 + \sum_{k=N+1}^{\infty} |v_k(t)|^2$$
$$\le \sum_{k=1}^{N} \left(\frac{|y_k^0 - x_k^0|^2}{\xi_1} + 2\left\langle \frac{y_k^0 - x_k^0}{\sqrt{\xi_1}} \right\rangle + |v_k(t)|^2 \right)$$
$$+ \sum_{k=N+1}^{\infty} |v_k(t)|^2. \tag{12}$$

Since,

$$\langle x, y \rangle = |x||y|,$$

We have

$$\sum_{k=1}^{\infty} |U_k(t,x,y,v)|^2 \le \sum_{k=1}^{N} \frac{|y_k^0 - x_k^0|^2}{\xi_1} + 2\sum_{k=1}^{N} \frac{|y_k^0 - x_k^0|}{\sqrt{\xi}_1} \sum_{k=1}^{N} |v_k(t)| + \sum_{k=1}^{N} |v_k(t)|^2 + \sum_{k=N+1}^{\infty} |v_k(t)|^2$$

From the fact that the

$$\sum_{k=1}^{N} \frac{|y_k^0 - x_k^0|^2}{(\rho - \sigma)^2} \le \sum_{k=1}^{\infty} \frac{|y_k^0 - x_k^0|^2}{(\rho - \sigma)^2},$$

we get

$$\sum_{k=1}^{\infty} |U_k(t,x,y,v)|^2 \le \sum_{k=1}^{\infty} \frac{|y_k^0 - x_k^0|^2}{\xi_1} + 2\sum_{k=1}^{\infty} \frac{|y_k^0 - x_k^0|}{\sqrt{\xi}_1} \sum_{k=1}^{\infty} |v_k(t)| + \sum_{k=1}^{\infty} |v_k(t)|^2. \tag{13}$$

Using equation (10) and definition 1, we have from inequality (13)

$$\sum_{k=1}^{\infty} |U_k(t,x,y,v)|^2 \le (\rho - \sigma)^2 + 2(\rho - \sigma) \sum_{k=1}^{\infty} |v_k(t)| + \sigma^2$$
$$\le (\rho - \sigma)^2 + 2(\rho - \sigma)\sigma + \sigma^2 = \rho.$$

Since

$$\sum_{k=1}^{\infty} |v_k(t)| \le \left(\sum_{k=1}^{\infty} |v_k(t)|^2 \right)^{\frac{1}{2}} \le \sigma,$$

and strategy (11) is admissible.

A_3: Next, it is not difficult to prove that $x_k(\xi) = y_k(\xi)\ k$ for someξ. Using admissible strategy and solution of the simple motion (5), we have

$$x_k(\xi) = x_k^0 + \int_0^{\xi} U_k(t,x,y,v)dt = x_k^0 + \int_0^{\sqrt{\xi_1}} U_k(t,x,y,v)dt + \int_{\sqrt{\xi_1}}^{\xi} U_k(t,x,y,v)dt\,.$$

Suppose that the pursuer uses the strategy (11) and in accordance with definition 2 & 4 from the above equation, we get

$$x_k(\xi) = x_k^0 + \int_0^{\sqrt{\xi_1}} \left(\frac{y_k^0 - x_k^0}{\sqrt{\xi_1}} + v_k(t)\right)dt + \int_{\sqrt{\xi_1}}^{\xi} v_k(t)dt$$

$$= x_k^0 + (y_k^0 - x_k^0)\int_{\xi}^{\sqrt{\xi_1}} \frac{1}{\sqrt{\xi_1}}dt + \left[\int_0^{\sqrt{\xi_1}} \int_{\sqrt{\xi_1}}^{\xi} (v_k(t)dt)\right]$$

$$= y_k^0 + \int_0^{\xi} v_k\,(t)dt = y_k(\xi).$$

Consequently, $x_k(\xi) = y_k(\xi)$. This completes the proof of part 1.

Theorem 2 *Let $\rho > \sigma$, then $x_k(\xi) = y_k(\xi), \xi > 0$ from initial position$x^0 = (x_1^0,\ x_2^0, \dots), y^0 = (y_1^0,\ y_2^0, \dots)$ in differential games Ω_2 for some timeξ.*

Proof: Similarly, we prove the theorem into three steps, firstly, is to set the strategy. Secondly, we show that the strategy is admissible and prove that the game is completed.

B_1: We construct the strategy of the pursuer(ρ) for which control function satisfy integral constraint.

$$U_k(t,x,y,v) = \begin{cases} \dfrac{y_k^0 - x_k^0}{\xi_1} + v_k(t) & 0 \le t \le \xi_1 \\ v_k(t), & t > \xi_1, \end{cases} \quad k = 1,2,\dots \tag{14}$$

B_2: In a similar fashion as in part 1 above, it is clear that $U(.,v(.)) \epsilon\ \psi_2(P)$, we show that strategy (14) is admissible. We have

$$\sum_{k=1}^{\infty}\int_0^{\xi} |U_k(t,x,y,v)|^2 = \sum_{k=1}^{\infty}\int_0^{\xi_1} |U_k(t,x,y,v)|^2 dt + \sum_{k=1}^{\infty}\int_{\xi_1}^{\xi} |U_k(t,x,y,v)|^2 dt$$

In accordance with definition 3 and strategy (14), we get

$$\sum_{k=1}^{\infty}\int_0^{\xi} |U_k(t,x,y,v)|^2 = \sum_{k=1}^{\infty}\int_0^{\xi_1} \left|\frac{y_k^0 - x_k^0}{\xi_1} + v_k(t)\ \right|^2 dt + \sum_{k=1}^{\infty}\int_{\xi_1}^{\xi} |v_k(t)|^2\,dt$$

$$\le \sum_{k=1}^{\infty}\int_0^{\xi_1} \left(\frac{|y_k^0 - x_k^0|^2}{\xi_1^2} + 2\left\langle\frac{y_k^0 - x_k^0}{\xi_1}\ v_k(t)\right\rangle + |v_k(t)|^2\right)dt$$

$$+ \sum_{k=1}^{\infty}\int_{\xi_1}^{\xi} |v_k(t)|^2\,dt$$

$$\le \sum_{k=1}^{\infty}\int_0^{\xi_1} \frac{|y_k^0 - x_k^0|^2}{\xi_1^2}dt + 2\sum_{k=1}^{\infty}\int_0^{\xi_1} \frac{|y_k^0 - x_k^0|}{\xi_1}dt \sum_{k=1}^{\infty}\int_0^{\xi_1} |v_k(t)|\,dt\ + \sigma^2$$

We apply Cauchy Schwartz inequality

$$\int_0^T |f_k(t)g_k(t)|\,dt \le \left(\int_0^T |f_k(t)|^2\,dt\right)^{1/2} \left(\int_0^T |g_k(t)|^2\,dt\right)^{1/2}.$$

Indeed,

$$\sum_{k=1}^{\infty}\int_0^{\xi_1} |v_k(t)|\,dt \le \left(\sum_{k=1}^{\infty}\int_0^{\xi_1} |v_k(t)|^2\,dt\right)^{1/2} \left(\sum_{k=1}^{\infty}\int_0^{\xi_1} 1^2\,dt\right)^{1/2} \le \sigma\sqrt{\xi_1} \qquad (15)$$

Using the right hand side of inequality (15), we have

$$\sum_{k=1}^{\infty}\int_0^{\xi} |u_k(t,x,y,v|^2\,dt \le \sum_{k=1}^{\infty}\frac{|y_k^0 - x_k^0|^2}{\xi_1} + 2\sum_{k=1}^{\infty}\frac{|y_k^0 - x_k^0|}{\xi_1}\sigma\sqrt{\xi_1} + \sigma^2$$
$$= (\rho-\sigma)^2 + 2\sum_{k=1}^{\infty}\frac{|y_k^0 - x_k^0|}{\sqrt{\xi_1}}\sigma + \sigma^2 \qquad (16)$$

Then by applying equation (10) to the right hand side of equality (16), we get

$$\sum_{k=1}^{\infty}\int_0^{\xi} |u_k(t,x,y,v|^2\,dt = (\rho-\sigma)^2 + 2(\rho-\sigma)\sigma + \sigma^2.$$

And so,

$$\sum_{k=1}^{\infty}\int_0^{\xi} |u_k(t,x,y,v|^2 = \rho^2$$

$\boldsymbol{B_2}$: Next we prove that $x_k(\xi) = y_k(\xi)$ for the timeξ. We get

$$x_k(\xi) = x_k^0 + \int_0^{\xi} U_k(t,x,y,v)dt$$
$$= x_k^0 + \int_0^{\xi_1} U_k(t,x,y,v)dt + \int_{\xi_1}^{\xi} U_k(t,x,y,v)dt \qquad (17)$$

Here we use the strategy (14) and definition 2 & 4 of equation (17) to obtain

$$x_k(\xi) = x_k^0 + \int_0^{\xi}\left(\frac{y_k^0 - x_k^0}{\xi_1} + v_k(t)\right)dt + \int_{\xi_1}^{\xi} v_k(t)dt$$
$$= x_k^0 + (y_k^0 - x_k^0)\int_0^{\xi_1}\frac{1}{\xi_1}dt + \left|\int_0^{\xi_1}\int_{\xi_1}^{\xi}(v_k(t)dt)\right|$$
$$= y_k^0 + \int_0^{\xi} v_k(t)dt = y_k(\xi)$$

Hence, $x_k(\xi) = y_k(\xi)$ at some timeξ, the proof of the theorem (2) is completed.

4.0 Numerical Example

Now, we present numerical example to demonstrate our result.

Example: Consider the differential game Ω_1 and Ω_2 with $\rho = 5, \sigma = 3$ ξ=16

Let the players control functions be subjected to geometric and integral constraints

$$\sum_{k=1}^{\infty}\int_0^{16} |u_k(s)|^2 ds \le 25,\ \sum_{k=1}^{\infty}\int_0^{16} |v_k(s)|^2 ds \le 9. \qquad (18),$$

$$\sum_{k=1}^{\infty}|u_k(t)|^2 \le 25, \sum_{k=1}^{\infty}|v_k(t)|^2 \le 9, \qquad 0 \le t \le 16, \tag{19}$$

Clearly, the hypothesis of the theorem is satisfied since $\rho = 5 > 3 = \sigma$.

Consequently, from the conclusion of theorem (1), we define the pursuer's admissible strategy for which the control function satisfied geometric constraint

$$U_k(t) = \begin{cases} \dfrac{y_k^0 - x_k^0}{\sqrt{16}} + v_k(t) & 0 \le t \le 16 \\ v_k(t), & t > 16 \end{cases} k = 1,2, \dots \tag{20}$$

Therefore, guarantee completion of pursuit time $\xi = 16$ as follows

$$x_k(16) = x_k^0 + \int_0^{16} U_k(t)dt = x_k^0 + \int_0^{\sqrt{16}} U_k(t)dt + \int_{\sqrt{16}}^{16} U_k(t)dt$$

$$= x_k^0 + \frac{(y_k^0 - x_k^0)}{\sqrt{16}} + \int_0^{\sqrt{16}} dt + \int_0^{16} v_k(t)dt$$

$$= y_k^0 + \int_0^{16} v_k(t)dt = y_k(16).$$

Thus, $x_k(16) = y_k(16)$.

On the other hand from the theorem (2) above, we set the pursuer's admissible strategy for which the control function satisfies integral constraint as follows

$$U_k(t) = \begin{cases} \dfrac{y_k^0 - x_k^0}{16} + v_k(t) & 0 \le t \le 16 \\ v_k(t), & t > 16 \end{cases} k = 1,2, \dots \tag{21}$$

Also, guarantees completion of pursuit time $\xi = 16$ will be given as

$$x_k(16) = x_k^0 + \int_0^{16} U_k(t)dt = x_k^0 + \int_0^{8} U_k(t)dt + \int_8^{16} U_k(t)dt$$

$$= x_k^0 + \frac{(y_k^0 - x_k^0)}{8} \int_0^{8} dt + \int_0^{16} v_k(t)dt$$

$$= y_k^0 + \int_0^{16} v_k(t)dt = y_k(16).$$

Consequently, $x_k(16) = y_k(16)$.

5.0 Conclusion

In this research work, a differential game of pursuit for an infinite system of simple motion in the plane is presented where the geometric and integral constraints are imposed on the control functions of the pursuer (P) and the evader (E).

In the game, we have investigated a condition of completion of the game and prove $x_k(\xi) = y_k(\xi)$ at some time ξ. In addition to the main result, an illustrated example is provided to justify the result.

On this note therefore, it is worthy to identify the following points for further research work:

i. The game(5) can be extended to many pursuer and one evader of the differential game problem in the plane

ii. Also, an optimal pursuit time for the system (5) can be studied in Hilbert Space.

Declarations

Ethics approval and consent to participate

Not Applicable

Consent for publication

All authors have read and consented to the submission of the manuscript.

Availability of data and material

Not Applicable.

Competing interests

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