



Computer simulation of transient electroosmotic flow of Oldroyd-B fluid in a micro-channel with Caputo fractional derivative and heat transfer

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Abstract	Article History
The Oldroyd-B fluid electroosmotic flow is described using the Caputo fractional derivative in a microchannel, where the Poisson-Boltzmann equation is used to describe the potential electric field applied along the length of the microchannel. By combining the Laplace transform with the classical method of partial differential equations and the finite Fourier sine transform, the component of the velocity profile of the equation is analytically solved. Laplace and finite Fourier sine transforms are used to solve the temperature component analytically. The inverse Laplace transform is obtained by the virtue of the Gave Stehfest technique, and Mathcad software is used for numerical simulation. As a memory effect, we see that the fluid velocity and temperature distribution diminish with increasing values of the fractional-order parameters. The graphical results are discussed in detail, demonstrating the impacts of time, relaxation time, retardation time, electro-kinetic width, and fractional parameters on the fluid velocity as well as the effects of time and fractional parameters on the microchannel temperature distribution. These are examined, and discovered that they are helpful for designing microfluidic devices. These tools could be helpful for diagnosing and analyzing biomedical conditions as well as for the clinical detection of bacteria and viruses in biological processes.	Received: 24/08/2025 Accepted: 29/10/2025 Published: 31/12/2025
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Nomenclature

$A_1 = L + L^T$ Rivlin-Ericksen tensor

D_t^α, D_t^β Caputo fractional derivative operator of order

α and β

E_x external electric field

e Fundamental charge or basic charge

I Identity tensor

$\bar{J}$ unit vector in the same direction of the velocity flow

k Electrokinetic width

$k^2 = \left(\frac{2\eta_0 z e^2 \psi}{\epsilon k_B T} \right)$ Debye-Hückel parameter

k_B Boltzmann constant

L velocity gradient

p pressure

R_ξ ratio of zeta-potential

S Extra-stress tensor or additional stress tensor

s Laplace transform variable

T Absolute temperature

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MATS – Mathematical Sciences

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t	Time
u	fluid velocity ms^{-1}
V	velocity vector
Z	Valence

Greek Letters

α, β	Fractional order derivative operators
∇	Gradient operator
ε	Dielectric permittivity
η_0	ion density
λ_1	Relaxation time
λ_2	Retardation time
μ	Dynamic viscosity
ξ	zeta potential
ξ_0	zeta shear rate velocity
ρ	fluid density
ρ_e	net charge density of the electrolyte solution,
C m^{-3}	
$-\rho I$	indeterminate stress
τ_{xy}	shear stress
Φ	Total electric potential
ϕ	Applied electric field
ψ	Electrical potential

1.0 Introduction

The emergence of Lab-on-a-Chip (LOC) technology in recent years has sparked researchers' interest in the behaviors of fluid flow, particularly when those behaviors are influenced by certain physical quantities like an electric field or a magnetic field. Recent developments in micro- and nanotechnology have given rise to a new discipline called LOC technology, which enables the integration of several components into a single micro-device. Researchers in a wide range of scientific and technical sectors have been interested in microfluidic devices. This is due to the outstanding solutions that lab-on-a-chip technology provides for highly specialized analysis in such fields like chemistry, biology, medicine, engineering, etc.

Electroosmotic flow is a method of moving fluids by applying an external electric field to an electrolyte solution that is close to a charge surface. This is one of the fundamental electro-kinetic phenomena that can be employed in a wide range of micro-device applications, including biological and chemical analysis, medical diagnostics, pumping, controlling, transporting, and manipulating fluid of nano-volumes [1, 2, 3]. An integrated micro-pump called electroosmotic flow is often produced by employing electrokinetic phenomena. This describes fluid

movement along electrostatically charged surfaces brought on by the introduction of an external electric field. This is a technologically advanced technique of transportation that is effective and has the potential to be used in a variety of industries by applying an external electric field across the microchannel. For instance, this way of fluid flow in a capillary annulus with high zeta potentials was taken into consideration by [4]. In [5] they examined the dynamic element of the electroosmotic aspect of fluid flow in a cylindrical micro-capillary, while [6] explored the flow through the annulus and discovered that the flow is in the opposite direction from the capillary but has the same sign of net charges. A cation-exchange membrane was used in an experiment by Barragan et al. in [7], who found that the fluid flow has peaked at two frequencies. The first was at a low frequency (Hz), and the second was at a frequency of the order of kHz. The authors of [8] explored the importance of chemical fraternization using the peristaltic electroosmosis conveyance method.

Researchers' interest in fractional calculus has increased recently due to its use in a variety of scientific, engineering, biological, and medical diagnostic fields [9, 10, 11]. Due to their numerous uses and versatility in characterizing the behaviors of non-Newtonian fluids in various disciplines of science and engineering, fractional derivatives have been the subject of several works (see, [12,13, 14, 15]), and as a result, shear influences viscosity, thinning, and thickening in non-Newtonian fluids. Due to its deformation-rate dependence and viscoelasticity feature, even blood flow in a tiny artery can exhibit non-Newtonian fluid behavior [16]. Recently in [17], solutions of the electroosmotic model flowing via a microchannel were obtained using numerical and semi-analytical approaches. Upon closer inspection, scientists discovered that the channel walls' electric double layer had certain charges that caused the velocity to become asymmetric. For an incompressible fractional order Oldroyd-B fluid model for both relaxation time and retardation time, Nadeem [18] used transformation methods to determine the velocity field within an infinite edge. He demonstrated the substantial contributions of the fractional Oldroyd-B model, oscillation type, and the theory of non-Newtonian fluids. Using Weber transformation, the authors of [19, 20] were able to arrive at exact analytical answers for Oldroyd-B fluid models. With the aid of the Laplace transform (LT) and finite Hankel transform (FHT), Jamil et al. [21] were able to tackle the issue of an unstable flow of a fractional derivative model of an Oldroyd-B fluid between two infinite coaxial cylinders. The authors examined the fractional calculus analysis of an Oldroyd-B fluid caused by an accelerating plate in [22, 23], where they were able to achieve a closed form solution for the velocity and

shear stress by using the fox H-function (LT). The study by [24] described the electroosmotic flow of a fractional order Oldroyd-B fluid in a circular microchannel. Zafar et al. [25] studied fractional derivative Oldroyd-B fluid unstable coquette flows. Shah et al. [26] studied electroosmotic fluid flow and included discussion of the effects of several influencing factors. As a result, the potential impacts of the relaxation time, fractional parameters, retardation time, and electro-kinetic width on the fluid flow velocity about the electro-kinetically driven flow transport in the microchannel are the main emphasis of this study. We also investigated how time and fractional parameters affect the temperature distribution in the microchannel, which may be significant for liquid transport in various containments and the disposal of chemical waste.

2.0 The Concepts of Fractional Calculus

Here, we give some fundamental definitions and significant aspects of fractional calculus that are applicable and stimulating at different levels of development and may also operate as a gateway to a unique body of knowledge. Because of its importance to the study of fractional calculus, the generalized function of Lorenzo-Hartley (LH) [27] in particular has drawn the attention of several scholars in the fields of science, engineering, and technology.

Definition 1: [27] The generalized LH function is defined by

$$G(t, d)_{a,b,c} = \sum_{j=0}^{\infty} \frac{\Gamma(1-c)(-d^k)t^{(c+j)-b-1}}{\Gamma(1+j)\Gamma(1-j-c)\Gamma((1+j)a-b)} \quad (1)$$

$$\left(\text{Re}(ac-b) > 0, \text{Re}(s) > 0, \left| \frac{d}{s^a} \right| < 1 \right)$$

where a, b and c are not constrained to be integers. The LT of the LH generalized function is given by

$$\mathcal{L}\{G(t, d)_{a,b,c}\} = \frac{s^a}{(s^a - d)^c}, \quad (2)$$

$$\left(\text{Re}(ac-b) > 0, \text{Re}(s) > 0, \left| \frac{d}{s^a} \right| > 0 \right)$$

and the inverse of the LT is given as

$$\mathcal{L}^{-1}\left(\frac{s^b}{(s^a - d)^c}\right) = G(t, d)_{a,b,c}, \quad (3)$$

$$(\text{Re}(ac-b) > 0)$$

Definition 2: [13]. The Caputo fractional (Cf) derivative given as

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(1-s)^\alpha} ds, \quad (4)$$

where $[a, b], b > a$ and $\alpha \in [0, 1]$. The LT of Cf derivative operator is given by;

$$\mathcal{L}\{{}^C D_t^\alpha f(t)\}(s) = s^\alpha \mathcal{L}\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} D^k f(0) \quad (5)$$

where $\mathcal{L}$ denotes the LT. But if $0 < \alpha \leq 1$, then Eqn. (5) becomes

$$\mathcal{L}\{{}^C D_t^\alpha f(t)\}(s) = s^\alpha \mathcal{L}\{f(t)\}(s) - s^{\alpha-1} f(0) \quad (6)$$

2.1 Mathematical model

2.1.1 Flow field

Mathematical model equation for transient fluid flow in a microchannel given by Escandon et al. [2] as,

$$\rho \frac{\partial u(y, t)}{\partial t} = - \frac{\partial \tau_{xy}}{\partial y} + \rho_e E_x \quad (7)$$

The height of the microchannel is H, length L and width W as shown in Figure 1. The system is assumed to show viscoelasticity. In Eqn. (7) ρ , τ_{xy} , ρ_e , E_x are the fluid density, the shear stress, the net charge density and the external electric field. The boundary conditions of Eqn. (7) are:

$$u(y, 0) = 0, 0 \leq y \leq H, \quad (8)$$

$$\frac{\partial u(y, 0)}{\partial t} = 0, 0 \leq y \leq H, \quad (9)$$

and the initial conditions are,

$$u(0, t) = 0, t \geq 0, \quad (10)$$

$$u(H, t) = 0, t \geq 0. \quad (11)$$

Further, the fluid flow is caused by the application of the electric charges E_x which is placed externally close to the walls of the microchannel as illustrated in Figure 1.

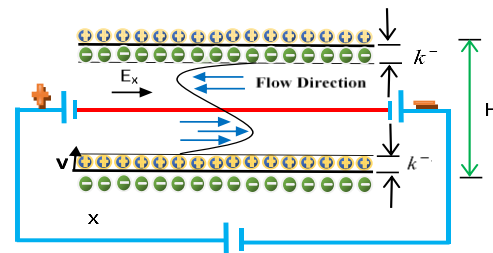


Figure 1: The geometry of the electroosmotic flow in a parallel microchannel for Oldroyd-B fluid

2.2 Governing equations for Oldroyd-B fluid flow

The constitutive equations of the unsteady Oldroyd-B fluid are given as: -

$$\tau = -\rho I + S, \quad \left(1 + \lambda_1^\alpha \frac{D^\alpha}{Dt^\alpha}\right) S = -\xi_0 \left(1 + \lambda_2^\beta \frac{D^\beta}{Dt^\beta}\right) \quad (12)$$

where p, I, ξ_0 and S are defined as the pressure, identity tensor, zero-shear rate viscosity and the additional stress tensor. The value λ_1 is the relaxation time and λ_2 is the retardation time. The fractional order in Eqn. (12) are denoted by α and β .

$$A_1 = L + L^T, \quad L = \text{grad} V, \quad (13)$$

$$\frac{D^\alpha}{Dt^\alpha} = D_1^\alpha S + (V \cdot \nabla) S - LS - SL^T, \quad (14)$$

$$\frac{D^\beta}{Dt^\beta} = D_1^\beta A_1 + (V \cdot \nabla) A_1 - LA_1 - A_1 L^T. \quad (15)$$

In the equations A_1 is the first Rivlin-Ericksen tensor.

The differential operators D_t^α and D_t^β are of order α , β with respect to time t , given in [2, 4, 5, 13]. Also $V = V(y, t) = u(y, t) \vec{j}$,
 (16)

where the velocity function $u(y, t)$ is in the y -direction. The unit vector $\vec{j}$ is in the direction of y . Putting Eqn. (12) into Eqn. (7) and by the use of the initial condition

$$S(y, 0) = 0; \quad y > 0, \quad (17)$$

we have

$$\left(1 + \lambda_1^\alpha D_t^\alpha\right) \tau(y, t) = -\xi_0 \left(1 + \lambda_2^\beta D_t^\beta\right) \frac{\partial u}{\partial y}(y, t) \quad (18)$$

where $\tau(y, t) = S_{xy}(y, t)$, and ξ_0 are as defined in Eqn. (12).

2.3 Electric Potential

The Poisson equation as defined by [8, 27, 28] is,

$$\frac{d^2 \Phi}{dx^2} + \frac{d^2 \Phi}{dy^2} = \frac{\rho_e}{\epsilon}. \quad (19)$$

In Eqn. (19), Φ is the total electric potential and ϵ is the dielectric permittivity. Eqn. (19) becomes

$$\Phi(x, y) = \phi(x) + \psi(y). \quad (20)$$

Putting Eqn. (20) into Eqn. (19), we have

$$\frac{d^2 \psi}{dy^2} = -\frac{\rho_e}{\epsilon}. \quad (21)$$

The Boltzmann distribution ρ_e in Eqn. (21) is given by:

$$\rho_e = -2z\eta_0 e \sinh\left(\frac{ze\psi}{k_B T}\right), \quad (22)$$

where z, η_0, e, k_B and T are the valence, the ion density, the basic charge, the Boltzmann constant and the absolute temperature. The top and bottom of the walls boundary conditions of Eqn. (21) are given respectively as:

$$\psi = \xi_1 \quad \text{at } y = 0 \quad (23)$$

and

$$\psi = \xi_2 \quad \text{at } y = H. \quad (24)$$

The Debye-Hückel approximation in [30, 31, 32] is defined by

$$\sinh\left(\frac{ze\psi}{k_B T}\right) \approx \left(\frac{ze\psi}{k_B T}\right) \text{ for } \left(\frac{ze\psi}{k_B T}\right) \ll 1.$$

Hence, Eqn. (21) becomes

$$\frac{d^2 \psi}{dy^2} = k^2 \psi \quad (25)$$

where k^2 is the parameter of Debye-Hückel and is defined by

$$k^2 = \left(\frac{2\eta_0 z^2 e^2}{\epsilon k_B T}\right).$$

The solution of Eqn. (25) subject to Eqn. (23) and Eqn. (24) is given by

$$\psi = \xi_1 \left\{ \left[\frac{R_\xi - e^{-kH}}{2 \sinh(kH)} \right] e^{ky} + \left[1 - \frac{R_\xi - e^{-kH}}{2 \sinh(kH)} \right] e^{-ky} \right\} \quad (26)$$

where

$$R_\xi = \frac{\xi_2}{\xi_1}, \quad \psi = k^2 \xi_1 \left(A e^{ky} + B e^{-ky} \right), \quad (27)$$

$$A = \frac{R_\xi - e^{-kH}}{2 \sinh(kH)}, \quad B = 1 - A, \quad (28)$$

$$\rho_e = -\epsilon k^2 \xi_1 \left(A e^{ky} + B e^{-ky} \right), \quad (29)$$

$$\rho_e E_x = -E_x \in k^2 \xi_1 \left(A e^{ky} + B e^{-ky} \right). \quad (30)$$

Substituting Eqn. (18) and Eqn. (30) into Eqn. (7) we have

$$\rho(1 + \lambda_1^\alpha D_t^\alpha) \frac{\partial}{\partial t} u(y, t) = \xi_0 (1 + \lambda_2^\beta D_t^\beta) \frac{\partial^2}{\partial y^2} u(y, t) - E_x \in k^2 \xi_1 (Ae^{ky} + Be^{-ky})$$

(31)

with Eqns. (8) - (11) as initial and boundary conditions. The non-dimensional parameters are:

$$\begin{aligned} \bar{y} &= \frac{y}{H}, \bar{u} = \frac{u}{u_{HS}}, \bar{t} = \frac{\xi_0 t}{\rho H^2}, \\ \bar{k}^2 &= (kH)^2, u_{HS} = \frac{-E_x \in \xi_1}{\xi_0}, \\ \bar{\lambda}_1 &= \frac{\lambda_1 \xi_0}{\rho H^2}, \bar{\lambda}_2 = \frac{\lambda_2 \xi_0}{\rho H^2} \end{aligned} \quad (32)$$

Using Eqn. (32) and Eqns. (8) - (11) and dropping the bar, Eqn. (31) reduces to

$$(1 + \lambda_1^\alpha D_t^\alpha) \frac{\partial}{\partial t} u(y, t) = (1 + \lambda_2^\beta D_t^\beta) \frac{\partial^2}{\partial y^2} u(y, t) + g(y) \quad (33)$$

where

$$g(y) = k^2 (Ae^{ky} + Be^{-ky}),$$

$$u(y, 0) = 0, \quad 0 \leq y \leq 1, \quad (34)$$

$$\frac{\partial}{\partial t} u(y, 0) = 0, \quad 0 \leq y \leq 1, \quad (35)$$

$$u(0, t) = 0, \quad t \geq 0, \quad (36)$$

$$u(1, t) = 0, \quad t \geq 0. \quad (37)$$

3.0 Exact solution of the velocity profile

3.1 LT together with finite sine Fourier transform

Using LT on Eqn. (33), Eqn. (36) and Eqn. (37) with the aid of Eqn. (34) and Eqn. (35), we have

$$(1 + \lambda_1^\alpha s^\alpha) s U(y, s) = (1 + \lambda_2^\beta s^\beta) \frac{\partial^2}{\partial y^2} U(y, s) + \frac{1}{s} g(y) \quad (38)$$

$$u(0, s) = 0, \quad s \geq 0, \quad (39)$$

$$u(1, s) = 0, \quad s \geq 0, \quad (40)$$

where $U(y, s)$ is the LT of $u(y, t)$.

Using Eqn. (38), we get

$$(1 + \lambda_1^\alpha s^\alpha) s V(n, s) = -(n\pi)^2 (1 + \lambda_2^\beta s^\beta) V(n, s) + \frac{1}{s} G(n), \quad (41)$$

where $V(n, s)$ denotes the finite sine Fourier transforms of $U(y, s)$. Similarly $G(n)$ is the finite sine Fourier transform of $g(y)$. Rewriting Eqn. (41) as

$$V(n, s) = G(n) F(n, s) \quad (42)$$

and

$$F(n, s) = \frac{1}{s [\lambda_1^\alpha s^{\alpha+1} + s + (n\pi)^2 + \lambda_2^\beta (n\pi)^2 s^\beta]}. \quad (43)$$

Rearranging Eqn. (43) gives: -

$$\begin{aligned} F(n, s) &= \frac{1}{\lambda_1^\alpha s^2 \left[s^\alpha + \frac{1}{\lambda_1^\alpha} + \frac{(n\pi)^2}{\lambda_1^\alpha} s^{-1} + \frac{\lambda_2^\beta (n\pi)^2}{\lambda_1^\alpha} s^{\beta-1} \right]} \\ &= P_0 s^{-2} \frac{1}{(s^\alpha + P_0) + (p_n s^{-1} + q_n s^{\beta-1})} \end{aligned} \quad (44)$$

where

$$P_0 = \frac{1}{\lambda_1^\alpha}, p_n = \frac{(n\pi)^2}{\lambda_1^\alpha}, q_n = \frac{\lambda_2^\beta (n\pi)^2}{\lambda_1^\alpha}.$$

Writing the functional Eqn. (44) in series form we have

$$\begin{aligned} F(n, s) &= P_0 s^{-2} \sum_{k=0}^{\infty} \frac{(-1)^k (p_n s^{-1} + q_n s^{\beta-1})^k}{(s^\alpha + P_0)^{k+1}} \\ &= P_0 s^{-2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(s^\alpha + P_0)^{k+1}} \sum_{m=0}^k \frac{k!}{m!(k-m)!} (p_n s^{-1})^m (q_n s^{\beta-1})^{k-m} \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{P_0 k! p_n^m q_n^{k-m}}{m!(k-m)!} \frac{s^{\beta(k-m)-k-2}}{(s^\alpha + P_0)^{k+1}}. \end{aligned} \quad (45)$$

Expressing the transformed solution of Eqn. (42) we get

$$V(n, s) = G(n) \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{P_0 k! p_n^m q_n^{k-m}}{m!(k-m)!} \frac{s^{\beta(k-m)-k-2}}{(s^\alpha + P_0)^{k+1}} \quad (46)$$

Using the LH generalized function we get,

$$G(t, d)_{a,b,c} = L^{-1} \left\{ \frac{s^b}{(s^a - d)^c} \right\}, \quad (47)$$

$$\text{Re}(ac - b) > 0,$$

with

$$\begin{aligned} a &= \alpha, \quad b = \beta(k-m) - k - 2, \\ c &= k + 1, \\ ac - b &= \alpha k + \alpha + \beta m + 2 + k(1 - \beta) > 0. \end{aligned} \quad (48)$$

With the help of Eqn. (47) and Eqn. (48), the inverse LT of Eqn. (46) is given by:

$$V(n, s) = G(n) \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{P_0 k! p_n^m q_n^{k-m}}{m!(k-m)!} G(t - P_0)_{\alpha, \beta(k-m)-k-2, k+1} \quad (49)$$

Using the inverse finite Fourier sine transform, Eqn. (49) reduces to

$$\begin{aligned} u(y, t) &= \\ 2 \sum_{n=1}^{\infty} \left\{ G(n) \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{P_0 k! p_n^m q_n^{k-m}}{m!(k-m)!} G(t, -P_0)_{\alpha, \beta(k-m)-k-2, k+1} \right\} \sin(n\pi y). \end{aligned} \quad (50)$$

3.2 Exact solution of the temperature profile

The equation for the temperature is given by

$$\rho c_p \frac{\partial}{\partial t} T(y, t) = k \frac{\partial^2}{\partial y^2} T(y, t) + \frac{\bar{J}\bar{J}}{\sigma_e} + \tau_{xy} l \quad (51)$$

subject to the initial and boundary conditions

$$T(y, 0) = T_w, \quad 0 \leq y \leq H, \quad (52)$$

$$T(H, t) = T_w, \quad t > 0, \quad (53)$$

where T_w is the wall temperature. Using the Oldroyd-B model in (18), where

$$l = \frac{\partial u}{\partial y}, \quad \bar{J} = \sigma_e \bar{E}, \quad \bar{E} = E_y \bar{e}_y$$

Eqn. (51) yields,

$$\begin{aligned} \rho c_p \left(1 + \lambda_1^\alpha D_t^\alpha \right) \frac{\partial}{\partial t} T(y, t) &= k \left(1 + \lambda_1^\alpha D_t^\alpha \right) \frac{\partial^2}{\partial y^2} T(y, t) \\ &+ \sigma_e E_y^2 + \xi_0 \left(1 + \lambda_2^\beta D_t^\beta \right) \left(\frac{\partial u}{\partial y} \right)^2 \end{aligned} \quad (54)$$

Using Eqn. (8) and the dimensionless temperature

$$\bar{T} = \frac{T - T_w}{T_1 - T_w}, \quad T_1 = \frac{\sigma_e H^2 E_y^2}{k}$$

we have, after dropping the bars,

$$\begin{aligned} P_r \left(1 + \lambda_1^\alpha D_t^\alpha \right) \frac{\partial}{\partial t} T(y, t) &= \left(1 + \lambda_1^\alpha D_t^\alpha \right) \frac{\partial^2}{\partial y^2} T(y, t), \\ &+ \lambda_2^\beta D_t^\beta g(y, t) + f(y, t) \end{aligned} \quad (55)$$

$$T(y, 0) = 0, \quad 0 \leq y \leq 1, \quad (56)$$

$$T(1, t) = 0, \quad t > 0, \quad (57)$$

where

$$f(y, t) = \delta_y + g(y, t), \quad \delta_y = \frac{H^2}{k T_0} \sigma_e E_y^2, \quad g(y, t) = P_r E_c \left(\frac{\partial u}{\partial y} \right)^2, \quad P_r E_c = \xi_0 \frac{u_{HS}^2}{k T_0}$$

so that

$$\left(\frac{\partial}{\partial y} u(y, t) \right)^2 = \quad (58)$$

$$\left[2n\pi \sum_{n=1}^{\infty} \left\{ G(n) \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{P_0 k! p_n^m q_n^{k-m}}{m!(k-m)!} G(t, -P_0)_{\alpha, \beta(k-m)-k-2, k+1} \right\} \cos(n\pi y) \right]^2$$

3.3 Laplace transform together with finite Fourier sine transform

Calculating the LT of Eqn. (55) and Eqn. (57) with the aid of Eqn. (56), we obtain

$$P_r \left(1 + \lambda_1^\alpha s^\alpha \right) \theta(y, s) = \left(1 + \lambda_1^\alpha s^\alpha \right) \frac{\partial^2}{\partial y^2} \theta(y, s) \quad (59)$$

$$\begin{aligned} &+ \lambda_2^\beta s^\beta G(y, s) + F(y, s) \\ \theta(1, s) &= 0, \quad s > 0, \end{aligned} \quad (60)$$

where

$$\theta(y, s) = \int_0^\infty T(y, t) e^{-st} dt,$$

$$G(y, s) = \int_0^\infty g(y, t) e^{-st} dt$$

and

$$F(y, s) = \int_0^\infty f(y, t) e^{-st} dt$$

are the LT of $T(y, t)$, $g(y, t)$ and $f(y, t)$, respectively.

Taking the LT of

$$\begin{aligned} G(y, s) &= \\ \int_0^\infty \left[2n\pi \sum_{n=1}^{\infty} \left\{ G(n) \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{P_0 k! p_n^m q_n^{k-m}}{m!(k-m)!} G(t, -P_0)_{\alpha, \beta(k-m)-k-2, k+1} \right\} \cos(n\pi y) \right]^2 e^{-st} dt \end{aligned} \quad (61)$$

we obtain the following

$$\begin{aligned} G(y, s) &= \left[2n\pi \sum_{n=1}^{\infty} \left\{ G(n) \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{P_0 k! p_n^m q_n^{k-m}}{m!(k-m)!} \right\} \cos(n\pi y) \right]^2 \\ &\times \int_0^\infty \left[G(t, -P_0)_{\alpha, \beta(k-m)-k-2, k+1} \right]^2 e^{-st} dt, \end{aligned} \quad (62)$$

$$F(y, s) = \int_0^\infty [\delta_y + g(y, t)] e^{-st} dt, \quad (63)$$

$$F(y, s) = \frac{1}{s} \frac{H^2}{kT_0} \sigma_e E_y^2 + G(y, s). \quad (64)$$

Calculating the finite Fourier sine transform of Eqn. (59), we get

$$P_r(1 + \lambda_1^\alpha s^\alpha) s \varphi(n, s) = - (n\pi)^2 (1 + \lambda_1^\alpha s^\alpha) \varphi(n, s) + \lambda_2^\beta s^\beta H(n, s) + h(n, s) \quad (65)$$

where

$$\varphi(n, s) = \int_0^1 \theta(y, s) \sin(n\pi y) dy$$

$$H(n, s) = \int_0^1 G(y, s) \sin(n\pi y) dy$$

and

$$h(n, s) = \int_0^1 F(y, s) \sin(n\pi y) dy$$

are the finite sine Fourier transform of $\theta(y, s)$, $G(y, s)$ and $F(y, s)$, respectively.

$$P_r(1 + \lambda_1^\alpha s^\alpha) s \varphi(n, s) + (n\pi)^2 (1 + \lambda_1^\alpha s^\alpha) \varphi(n, s) = \lambda_2^\beta s^\beta H(n, s) + h(n, s) \quad (66)$$

$$\left[P_r(1 + \lambda_1^\alpha s^\alpha) s + (n\pi)^2 (1 + \lambda_1^\alpha s^\alpha) \right] \varphi(n, s) = \lambda_2^\beta s^\beta H(n, s) + h(n, s) \quad (67)$$

$$\varphi(n, s) \left[s + \lambda_1^\alpha s^{\alpha+1} + \frac{(n\pi)^2}{P_r} + \frac{(n\pi)^2}{P_r} \lambda_1^\alpha s^\alpha \right] = \frac{\lambda_2^\beta s^\beta H(n, s) + h(n, s)}{P_r} \quad (68)$$

$$\varphi(n, s) = \frac{\lambda_2^\beta s^\beta H(n, s) + h(n, s)}{P_r \left[s + \lambda_1^\alpha s^{\alpha+1} + \frac{(n\pi)^2}{P_r} + \frac{(n\pi)^2}{P_r} \lambda_1^\alpha s^\alpha \right]} \quad (69)$$

Rewriting Eqn. (69) we get,

$$\varphi(n, s) = F(n, s) [\lambda_2^\beta s^\beta H(n, s) + h(n, s)] \quad (70)$$

and

$$F(n, s) = \frac{1}{P_r \left[s + \lambda_1^\alpha s^{\alpha+1} + \frac{(n\pi)^2}{P_r} + \frac{(n\pi)^2}{P_r} \lambda_1^\alpha s^\alpha \right]} \quad (71)$$

Rearranging Eqn. (71) yields,

$$F(n, s) = \frac{1}{\lambda_1^\alpha P_r s \left[s^\alpha + \frac{s}{\lambda_1^\alpha} + \frac{(n\pi)^2}{\lambda_1^\alpha P_r} s^{-1} + \frac{(n\pi)^2}{\lambda_1^\alpha P_r} \lambda_1^\alpha s^{\alpha-1} \right]} = \frac{P_0 s^{-1}}{P_r} \frac{1}{(s^\alpha + P_0) + (P_1 s^{-1} + P_2 s^{\alpha-1})} \frac{1}{P_r} \quad (72)$$

where

$$P_0 = \frac{1}{\lambda_1^\alpha}, \quad P_1 = \frac{(n\pi)^2}{\lambda_1^\alpha}, \quad P_2 = (n\pi)^2$$

Rewriting the function $F(n, s)$ in series form we have

$$F(n, s) = \frac{P_0 s^{-1}}{P_r} \sum_{k=0}^{\infty} \frac{(-1)^k (P_1 s^{-1} + P_2 s^{\alpha-1})^k P_r^k}{(s^\alpha + P_0)^{k+1}} = \frac{P_0 s^{-1}}{P_r} \sum_{k=0}^{\infty} \frac{(-1)^k P_r^k}{(s^\alpha + P_0)^{k+1}} \sum_{m=0}^k \frac{k!}{m!(k-m)!} (P_1 s^{-1})^m (P_2 s^{\alpha-1})^{k-m} = \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-1)^k P_r^{k-1} k! P_0 P_1^m P_2^{k-m}}{m!(k-m)!} \frac{s^{\alpha(k-m)-k-1}}{(s^\alpha + P_0)^{k+1}} \quad (73)$$

The transformed solution of Eqn. (50) can be expressed as

$$\varphi(n, s) = \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-1)^k P_r^{k-1} k! P_0 P_1^m P_2^{k-m}}{m!(k-m)!} \frac{s^{\alpha(k-m)-k-1}}{(s^\alpha + P_0)^{k+1}} [\lambda_2^\beta s^\beta H(n, s) + h(n, s)] \quad (74)$$

$$\varphi(n, s) = \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-1)^k P_r^{k-1} k! P_0 P_1^m P_2^{k-m}}{m!(k-m)!} \left[\frac{\lambda_2^\beta s^{\alpha(k-m)+\beta-k-1} H(n, s)}{(s^\alpha + P_0)^{k+1}} + \frac{s^{\alpha(k-m)-k-1} h(n, s)}{(s^\alpha + P_0)^{k+1}} \right] \quad (75)$$

Taking the inverse LT of (74) we have

$$\varphi(n, s) = \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-1)^k P_r^{k-1} k! P_0 P_1^m P_2^{k-m}}{m!(k-m)!} [\theta(n, t) + \mathcal{G}(n, t)] \quad (76)$$

where

$$\theta(n, t) = \int_0^t \left[G(\lambda_2^\beta u, -P_0)_{\alpha, \alpha(k-m)+\beta-k-1, k} H(n, t-u) \right] du \quad (77)$$

and

$$\mathcal{G}(n, t) = \int_0^t \left[G(u, -P_0)_{\alpha, \alpha(k-m)-k-1, k} \right] h(n, t-u) du. \quad (78)$$

By the help of the LH generalized function, and convolution theorem, we get

$$G(t, d)_{a, b, c} = L^{-1} \left\{ \frac{s^b}{(s^a - d)^c} \right\}, \quad (79)$$

$$\operatorname{Re}(ac - b) > 0$$

with

$$\begin{aligned} a &= \alpha, \quad b = \alpha(k - m) + \beta - k - 1, \\ c &= k + 1 \\ ac - b &= \alpha(1 - m) - \beta + 1 > 0. \end{aligned} \quad (80)$$

For $\theta(\bar{n}, t)$ and we apply the inverse finite sine Ft to Eqn. (75) to get

$$T(\bar{y}, \bar{t}) = 2 \sum_{n=1}^{\infty} \left\{ \sum_{k=0}^{\infty} \sum_{m=0}^k (-1)^k p_r^{k-1} \frac{p_0 k! p_n^m q_n^{k-m}}{m! (k-m)!} \right\} \sin(n\pi\bar{y}) \quad (81)$$

4.0 The results obtained and discussion

The dimensionalized set of parameters were calculated and simulated using Mathcad software, and the graphical results are shown and explained here. However, before we move on to the proper analysis and discussion, we should note that the dimensionless parameters used here were taken from Escandon et al.

[2] and are as follows: $0.1 \leq H \leq 15 \mu m$,

$$1 \leq k^{-1} \leq 300 nm, \quad \rho \sim 10^3 kg m^{-3}, \quad \eta_0 \sim 10^3 Pa s,$$

$$E_x \leq 10^4 V m^{-1}, \quad |\xi| \leq 0.025 V \quad \text{and}$$

$$\epsilon \sim 7 \times 10^{-10} CV^{-1} m^{-1}.$$

Moreover, the range of parameter values is taken into account to study their influence on the flow characteristics as researched by Escandon et al. [2].

Most velocity graphs have been set at $\lambda_1 = 0.05$, $\lambda_2 = 0.03$, $k = 2$ for the sake of fair comparison so that two issues have the same setup.

The results demonstrate the influences of the fractional parameters (α, β), relaxation parameter λ_1 , retardation time λ_2 , electrokinetic width k , microchannel, and the ratio of zeta-potential R_ξ on the velocity as well as the influence of the fractional parameters (α, β) on the

temperature. Additionally, positive and negative zeta potential are taken into account in this work to demonstrate the behavior of the Oldroyd-B fluids and the impacts of memory on the fluid. Therefore, we regard all of the characteristics and variables mentioned above as dimensionless.

The dimensionless velocity profile $u(y, t)$ against the dimensionless coordinate (y) is shown in Figures 2(a)–(b) with fixed values for the fractional parameters (α, β) at various time values. The fluid flow's velocity profile is shown in the figure at four distinct times ($t = 0.01$, $t = 0.03$, $t = 0.05$, and $t = 0.10$) with $\lambda_1 = 0.05$ and $\lambda_2 = 0.03$ as well as the electro-kinetic width ($k = 2$). Figure 2(a) displays the value of the $\alpha > \beta$, and

Figure 2(b) displays the value of $\alpha < \beta$. As can be observed, due to the forces present in the electric double layer, for small values of time (t), the velocity increases rapidly from zero and achieves its maximum value only within the electric double layer before falling back to zero toward the other end of the region of the microchannel. Due to the fluid's inertia in this situation, the movement outside the electric double layer is delayed, and the hydrodynamic stress has not yet been transferred.

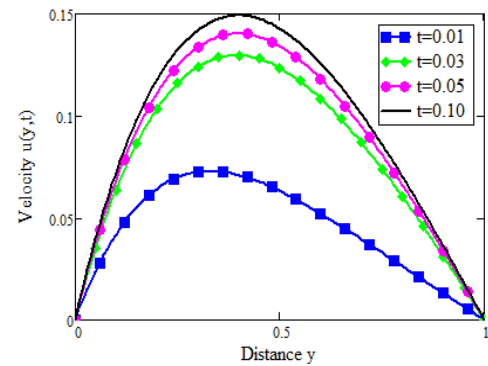
Figure 3 shows the fluid velocity with the fractional derivatives flow exactly like the one in Figure 2 graphs fluid model at the first stage when time is extremely small as in Figure 2, that is, $t = 0.01$, $t = 0.03$, $t = 0.5$ and $t = 0.10$. The velocity for an integer order fluid is higher than a fractional order derivative fluid for a longer period of time ($t = 0.10$). This is a result of the system's stability, which could improve with time. Both fractional and integer order derivative fluid models show that fluid velocity generally increases as time t increases. The magnitude of the velocity is greater, particularly when the ratio of zeta potential is negative than when the ratio of zeta potential is positive, as seen in Figure 3. Figures 4 and 5 display the velocity measured at the microchannel's midpoint (at $y = 0.5$) to demonstrate the influence of λ_2 on fluid flow, where the magnitude of the velocity decreases as λ_2 increases and decreases for ($\alpha < \beta$). Additionally, the velocity magnitude is larger when the ratio of zeta potential (R_ξ) is negative than when it is positive as shown in Figure 4 and Figure 5 respectively. The Figures demonstrate the effects of the fractional parameters (α, β) and the retardation time λ_2 . Figure 4 shows that for both values of $\alpha > \beta$ and $\alpha < \beta$, the retardation is greater for the fractional order model fluid than the integer order fluid. The fractional order $\alpha > \beta$ or $\alpha < \beta$ which improved fluid flow when the ratio

of zeta potential is negative, is shown graphically in a similar way as in Figure 5.

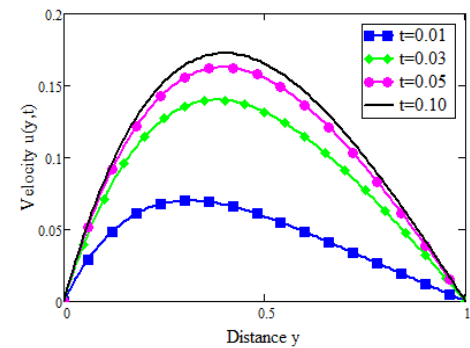
Figures 6 and 7 show electro-kinetic width for various values of the fractional parameters. These Figures illustrate how the electro-kinetic width of the micro-channel affects the dimensionless velocity. In both situations, when $\alpha > \beta$ and $\alpha < \beta$, the magnitude of the dimensionless velocity increases as the electro-kinetic width of the micro-channel grows. As depicted in Figure 7, the electro-kinetic breadth increases linearly as the fractional parameter increases and diminishes slowly but noticeably as the fractional parameter increases. Figures 8 and 9 demonstrate the behavior of transient dimensionless velocity evaluated at the channel's center, demonstrating that the classical fluid ($\alpha = 1$) exhibits the same pattern of flow whether $\alpha > \beta$ or $\alpha < \beta$ it is moving faster than the fractional modeled fluid. As a result, Figure 8 shows that the velocity initially exhibits minimum values close to the border whereas, following a rise in some fractional parameter values, the velocity eventually reaches a maximum value close to the boundary.

Similar outcomes are obtained from Figure 9, which showed the velocity field for the two fractional parameter values and when both parameters (y, t) are simultaneous variables. However, when $\alpha > \beta$ and when $\alpha < \beta$, the fractional velocity likewise starts slowly and moves back towards the steady state. However, it also starts steadily and moves away from the steady state. The Figures show that when time t is sufficiently large, the flow approaches the steady state. Similar Figures were provided in [25] by the authors in Figures 3 and 9 of their study. Thus, in Figure 3, they simply looked at the effects of α and β versus time (t) on the velocity, whereas in Figure 9, they looked at the impacts of λ_1 and λ_2 versus time (t) on the velocity and discovered that for particular values of the time t , the highest value of velocity occurs close to the boundary. The magnitude of the dimensionless velocity in Figures 10 and 11 are large for the smallest fractional parameter, and as time passes, the magnitude of the classical fluid ($\alpha = 1$) also grows. However, in both cases of the fractional parameters $\alpha > \beta$ or $\alpha < \beta$, the velocity of the classical fluid is larger than the velocity of the fractional fluid at any value of $t > 0.3$.

The fractional flow lowers steadily toward the top of the microchannel in Figure 10 for a small value of time (t), moving with increasing time towards the middle of the microchannel. The velocity is sharply higher towards the bottom of the microchannel. Figure 11 displays a pattern that is almost identical to that in Figure 10, which causes the fractional velocity to spread and the graphs formed dome sharp as depicted in the graphs of the Figure.

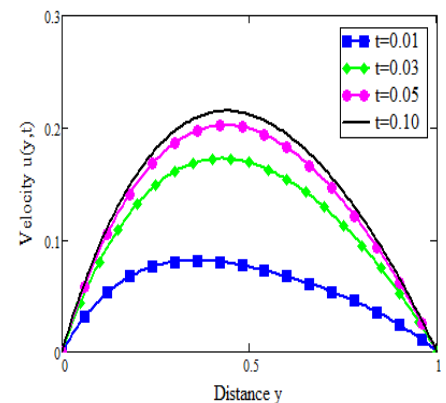


(a)

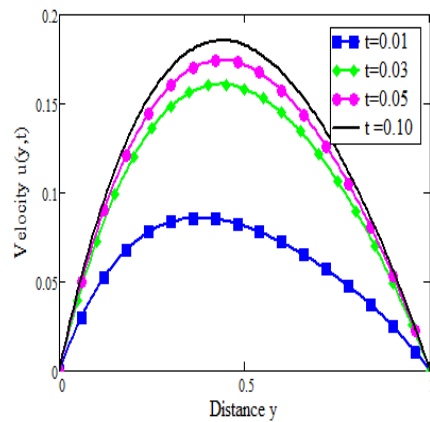


(b)

Figure 2: Velocity for Oldroyd-B fluid using positive ratio of zeta potential and time different $t = 0.01, 0.03, 0.05$ and 0.10 and $k = 2, \lambda_1 = 0.05, \lambda_2 = 0.03$ (a) $\alpha > \beta$ (b) $\alpha < \beta$

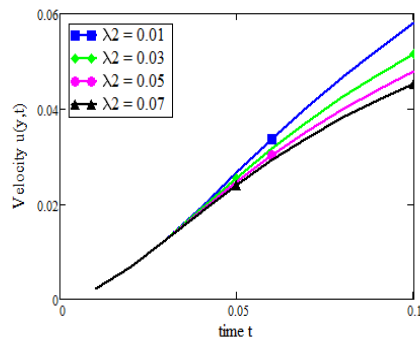


(a)

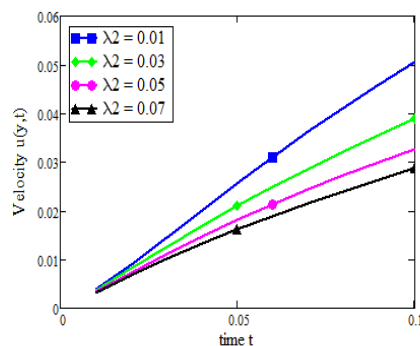


(b)

Figure 3: Velocity for Oldroyd-B fluid for negative ratio of zeta potential and time different $t = 0.01, 0.03, 0.05$ and 0.10 and $k = 2, \lambda_1 = 0.05, \lambda_2 = 0.03$ (a) $\alpha > \beta$ (b) $\alpha < \beta$

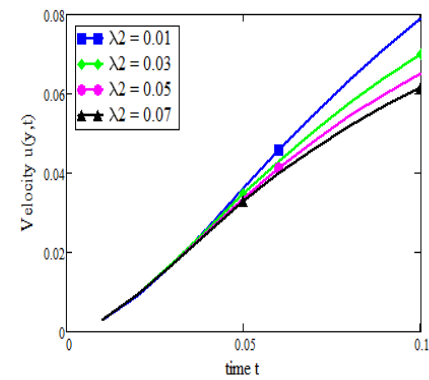


(a)

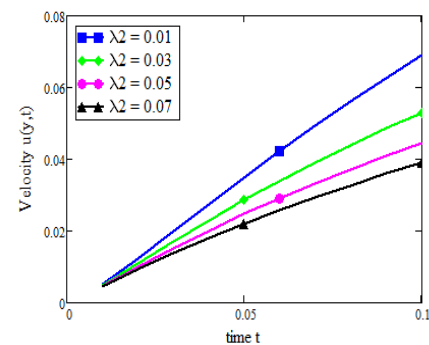


(b)

Figure 4: Effect of retardation at the centre of the microchannel for positive ratio of zeta potential $k = 2, y = 0.5, \lambda_1 = 0.05, \lambda_2 = 0.03$ (a) $\alpha > \beta$ (b) $\alpha < \beta$

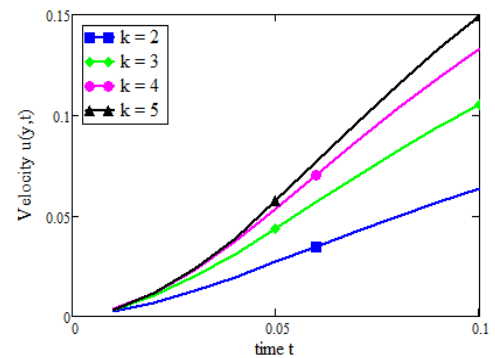


(a)

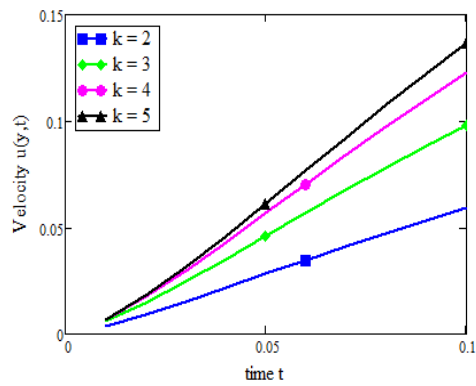


(b)

Figure 5: Effect of dimensionless retardation time on the fluid flow at the centre of the microchannel with negative ratio of zeta potential $k = 2, y = 0.5, \lambda_1 = 0.05, \lambda_2 = 0.03$ (a) $\alpha > \beta$ (b) $\alpha < \beta$

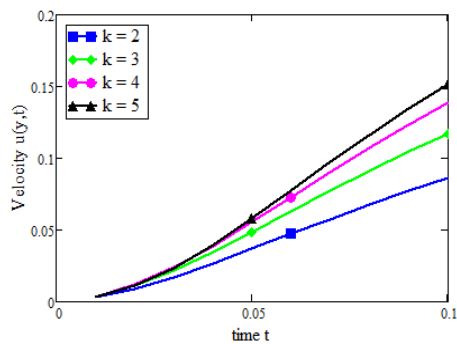


(a)

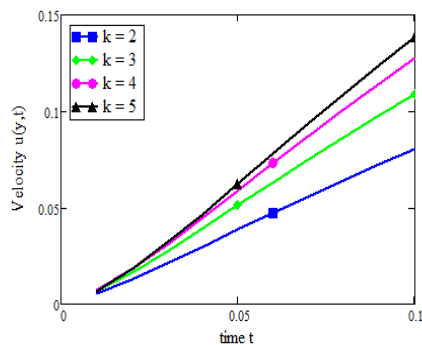


(b)

Figure 6: Velocity at the centre of the microchannel for various values of the electrokinetic width ($k = 2, 3, 4$ and 5) with positive ratio of zeta potential $y = 0.5, \lambda_1 = 0.05, \lambda_2 = 0.03$ (a) $\alpha > \beta$ (b) $\alpha < \beta$

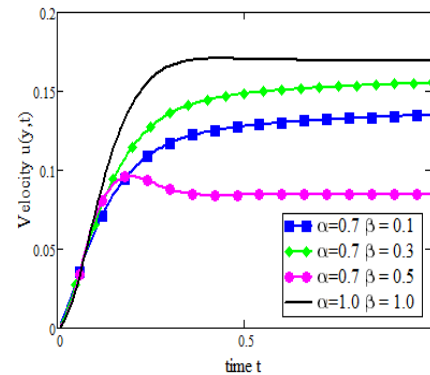


(a)

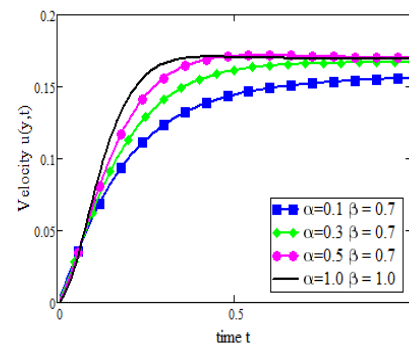


(b)

Figure 7: Velocity at the centre of the microchannel for some values of the electrokinetic width ($k = 2, 3, 4$ and 5) with negative ratio of zeta potential $y = 0.5, \lambda_1 = 0.05, \lambda_2 = 0.03$ (a) $\alpha > \beta$ (b) $\alpha < \beta$

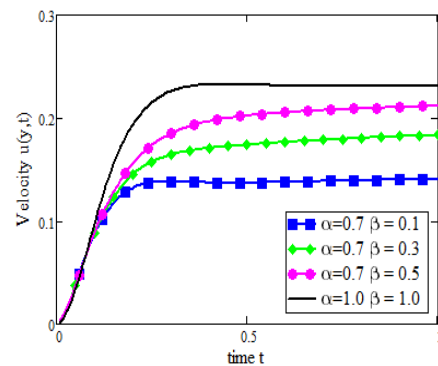


(a)

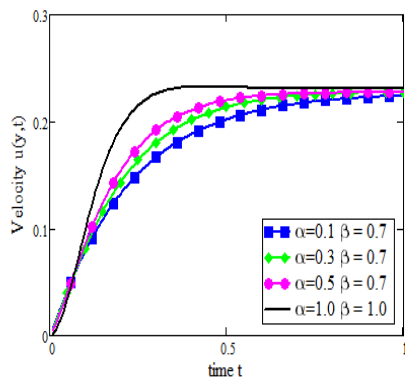


(b)

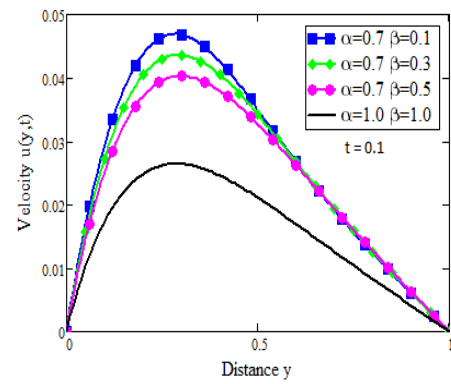
Figure 8: Velocity at the centre of the microchannel with positive ratio of zeta potential $k = 2, y = 0.5, \lambda_1 = 0.05, \lambda_2 = 0.03$ (a) $\alpha > \beta$ (b) $\alpha < \beta$



(a)

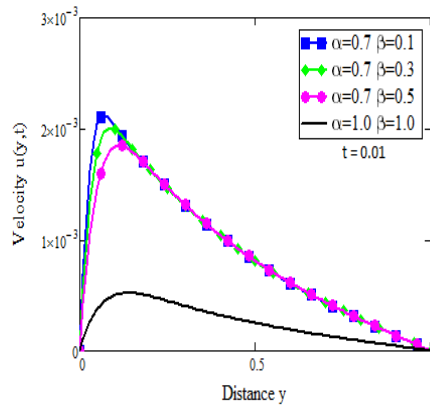


(b)

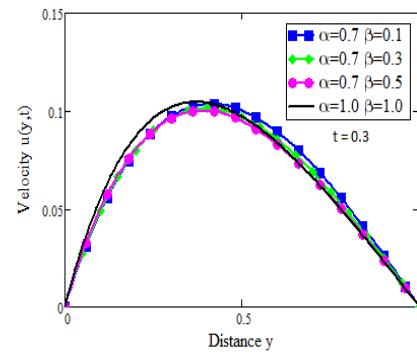


(c)

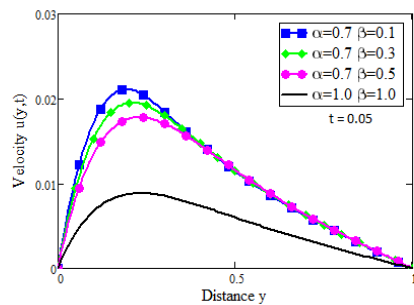
Figure 9: Velocity at the centre of the microchannel with negative ratio of zeta potential $k = 2$, $y = 0.5$, $\lambda_1 = 0.05$, $\lambda_2 = 0.03$ (a) $\alpha > \beta$ (b) $\alpha < \beta$



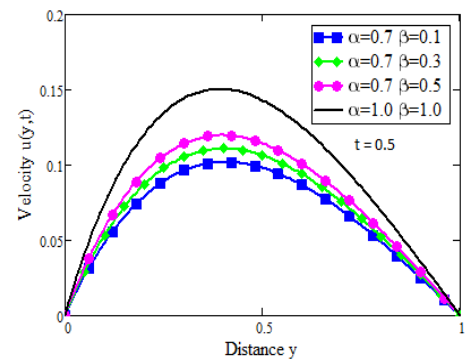
(a)



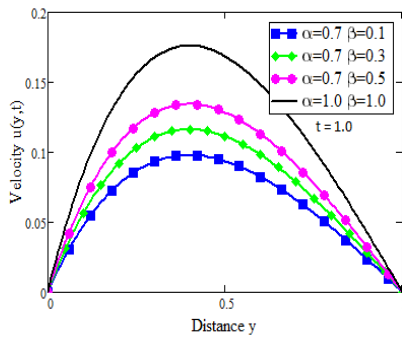
(d)



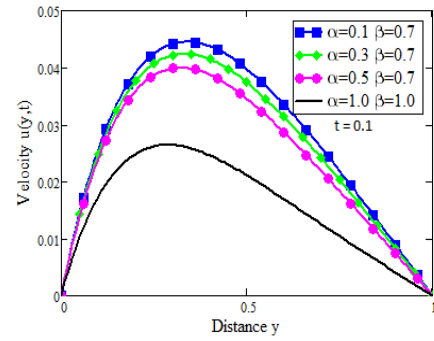
(b)



(e)

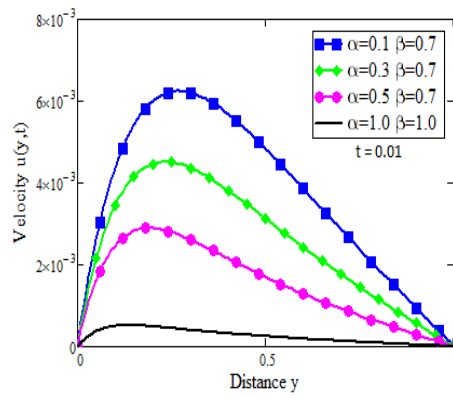


(f)

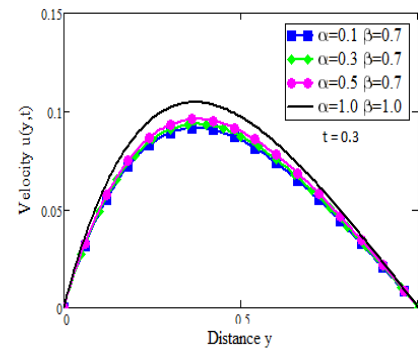


(c)

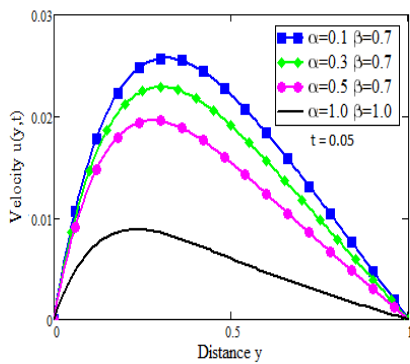
Figure 10: Velocity $u(y,t)$ evaluated at various values of time t when $\alpha > \beta$, $\lambda_1 = 0.05$, $\lambda_2 = 0.03$, $t = 0.1$ to 1.0



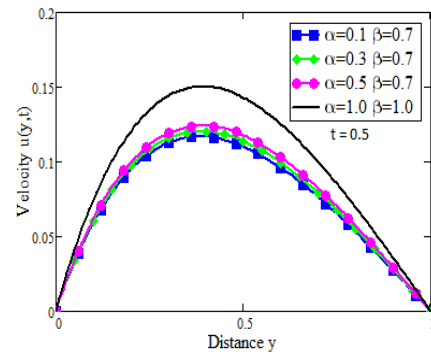
(a)



(d)



(b)



(e)

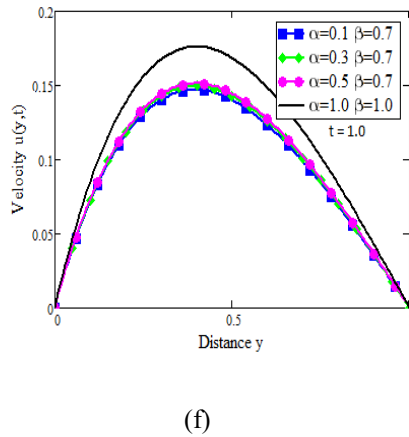


Figure 11: Velocity $u(y,t)$ evaluated at different value of time t when $\alpha < \beta$, $\lambda_1 = 0.05$, $\lambda_2 = 0.03$, $t = 0.1$ to 1.0

4.1 Temperature profiles

Temperature distribution is shown in Figure 12 with the fractional parameters held constant and the time varied. Increasing the value of time causes the temperature to rise for both fractional parameters $\alpha > \beta$ or $\alpha < \beta$. The temperature will always fluctuate as the distance gets wider. It is clear from this Figure that for both $\alpha > \beta$, $\alpha < \beta$ and at the times $t = 0.01, 0.03, 0.05$, and 0.07 , the amplitude of the constant temperature oscillates near the microchannel's center. The graphs in the Figure appear to show a modest periodicity in the time t variation, and small values of the time t seems to have the smallest effects on the amplitude of the periodicity flow rate. Observation of Figure 13 demonstrates how the temperature at the center of the micro-channel increases as time passes. One can infer from this Figure that temperature increases over time as it moves from the centerline to the microchannel's wall and reaches its maximum value there. This makes sense when considering the temperature distribution in the previous Figure, which shows that the wall is where this parameter fluctuates the greatest. As a result, near the channel centerline and for low values of the retardation time λ_2 ($\lambda_2 = 0$), the temperature increases clearly with $y = 0.5$, $k = 2$, and $\lambda_1 = 0.05$. Figure 14 shows that there is sliding temperature fluctuation over a short period of time. In this case, for relatively small values of the dimensionless time, the velocity rapidly increases from zero and reaches a maximum value inside the electric double layer (EDL), then maintains a constant velocity with minimal fluctuations at the central region of the microchannel due to the driving forces that only exist inside the EDL, and then decreases once more to zero

toward the end of the microchannel. The authors in [33] also noticed and observed the same fluid behavior as in Figures 12 and 14, namely that when the applied electric field moves the fluid rapidly inside the double layer, the steady state flow could be achieved, and the velocity at the middle of the pipe became maximum, given a dome-shaped composite cone (stratovolcanoes) at the upper surface of the channel and tends to a level base graphs.

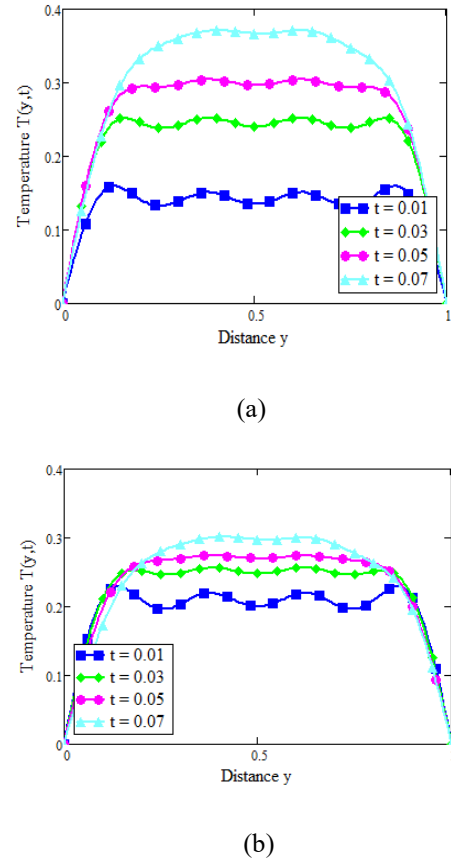


Figure 12: Temperature for fractional Oldroyd-B fluid with time variation $t = 0.01$ to 0.07 , $k = 2$, $\lambda_1 = 0.05$, $\lambda_2 = 0.03$, (a) $\alpha > \beta$ (b) $\alpha < \beta$

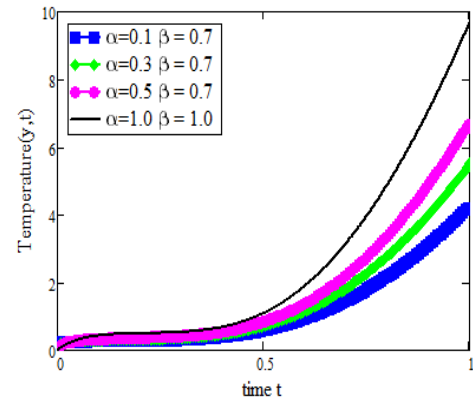
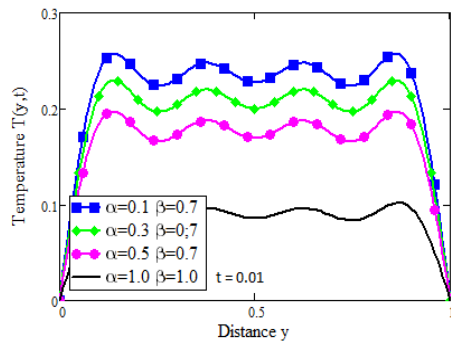
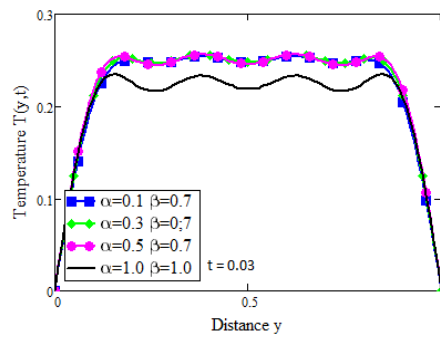


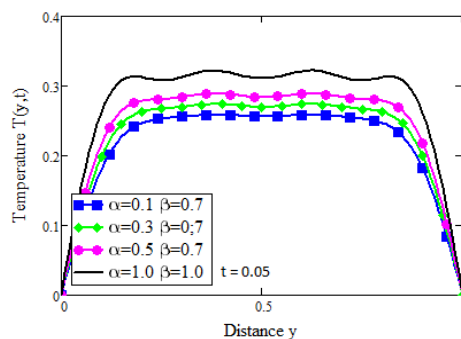
Figure 13: Temperature $T(y,t)$ evaluated at the centre of the microchannel with $k = 2, y = 0.5, \lambda_1 = 0.05$ and $\alpha < \beta$



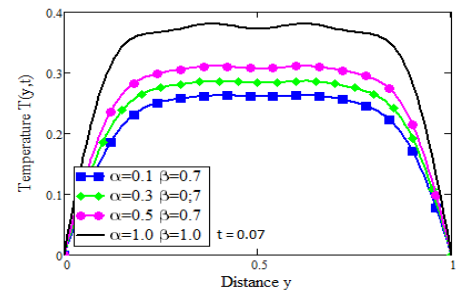
(a)



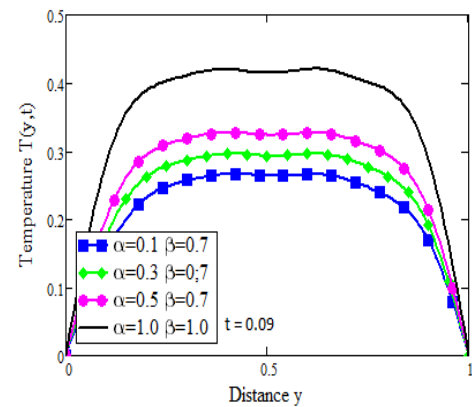
(b)



(c)



(d)



(e)

Figure 14: Temperature $T(y,t)$ evaluated at distance for different value of time t when $\alpha < \beta, \lambda_1 = 0.05, \lambda_2 = 0.03, t = 0.01$ -to- 0.09

5.0 Concluding Remarks

The goal of this paper is to use the Caputo time-fractional derivative to investigate the electro-osmotic flow of Oldroyd-B fluid in a microchannel under the Debye-Huckel approximation. Analytical solutions were found for the Poisson-Boltzmann equation, the equation of motion, and the constitutive equation of viscoelastic fluids using the electroosmotic force as the body force. The velocity field is computed using the Laplace and finite Hankel transforms. For the inverse Laplace transform we employed Gave Stehfest's algorithm. Following is a summary of the study's key findings as they are shown graphically:

- The study includes graphical illustration of how the electric field drives the fluid in the micro-channel in both fractional and classical motions.
- As time passes, the zeta potential and electroosmotic force rise in value, which causes the velocity profile to grow from the bottom wall to the middle of the microchannel for both fractional and classical flow fluids.

- The flow characteristics of the Oldroyd-B fluid appear to be similar to those of the comparable Maxwell fluids in the two classes of fluids (fractional and classical).
- Increase in the values of λ_2 and λ_1 result in the rise in the viscoelastic fluid, which increases the fluid's influence. Due to the need for sufficient time for the stress to react to deformation as a result, the fluid's velocity is reduced.
- Increasing the electrokinetic width value in the EDL at the beginning of time, increases the velocity. When k has a high value, this displays a thinner EDL.
- The fractional parameter significantly affects the fluid flow's velocity profile for small values of time.
- By appropriately altering their values, the fractional order derivatives regulate the velocity and temperature transmission in the microchannel.

The current study will be extremely helpful in these fields given the recent advancements in the study of micro-technologies, such as micro-fabrications and microfluidics, which both offer devices that could be used for analysis in bio-fluids, particularly in clinical, biological analysis, and chemical analysis.

For a future work, we believe that this work can be extended to Burger's fluids by including some relevant parameter, like the Burger's parameter. It is also possible to vary the influential parameters such as the fractional parameters, Reynolds number, magnetic parameter, radiation parameter, etc., and investigate their effects on the velocity and temperature.

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Declarations

Ethics approval and consent to participate

Not Applicable

Consent for publication

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Competing interests

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